Thursday seminar "Chiral kinetic theory"

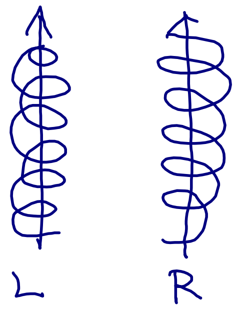
ref. Son & Yamamoto PRL 109, 181602 (2012) [1203.2697] cond-mat. mes-hall
PRD 87, 085016 (2013) [1210.8158] hep-th
Stephanov & Yin PRL 109. 162001 (2012) [1207.0747] hep-th
Stephanov, Yee & Yin PRD 91, 125014 (2015) [1501.00222] hep-th
Chang & Niu PRL 75, 1348 (1995) [cond-mat/9505021]

1. Introduction ··· chiral magnetic effect
2. derivation of chiral kinetic equation
3. Application / Consequences
4. Summary

1. Introduction

Fermions $\longrightarrow$ left-handed & right-handed (chirality)

$\Rightarrow$ in the massless limit, they are independent & conserved (classically)

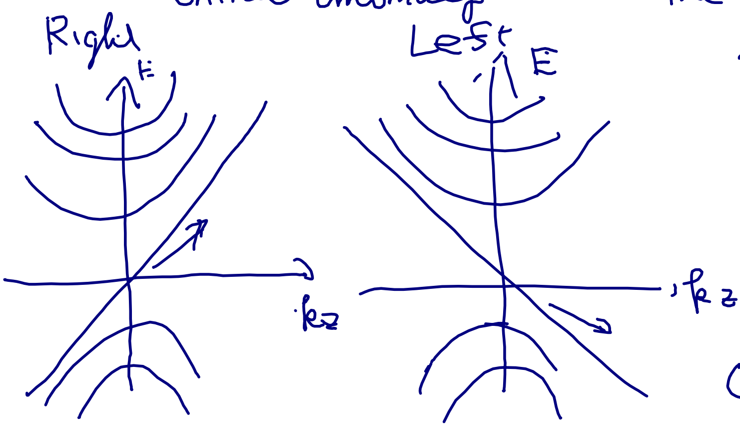


$$j^\mu_L \quad \& \quad j^\mu_R \qquad (\partial_\mu j^\mu_L = 0,\ \partial_\mu j^\mu_R = 0)$$

$\Rightarrow$ also convenient to define

$$j^\mu_V = j^\mu_R + j^\mu_L$$
$$j^\mu_A = j^\mu_R - j^\mu_L$$

If they are charged under a gauge symmetry, chiral anomaly breaks the conservation law



$$\partial_\mu j^\mu_A = -\frac{e^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{e^2}{2\pi^2} \mathbb{E}\cdot\mathbb{B}$$

$$(\partial_\mu j^\mu_V = 0)$$

(Nielsen & Ninomiya '83)

This effect is known to lead to the induced electric current

$$\mathbf{j}_{em} \ni \frac{e^2}{4\pi^2}\mu_A \mathbf{B}$$ "chiral magnetic effect"

cf. Ohm's current

$$\mathbf{j}_{em} = \sigma \mathbf{E}$$

→ e.g.) Modified Maxwell's eqn in MHD

$$\dot{\mathbf{B}} = -\nabla\times\mathbf{E} \qquad \nabla\times\mathbf{B} = \cancel{\dot{\mathbf{E}}} + \sigma\mathbf{E} + \frac{e^2}{4\pi^2}\mu_A \mathbf{B}$$

$$\hookrightarrow \dot{\mathbf{B}} = \frac{-1}{\sigma}\nabla\times\left(\nabla\times\mathbf{B} - \frac{e^2}{4\pi^2}\mu_A\mathbf{B}\right)$$

$$= \frac{1}{\sigma}\left(\nabla^2\mathbf{B} + \frac{e^2}{4\pi^2}\mu_A \nabla\times\mathbf{B}\right)$$

$$\hookrightarrow \frac{1}{\sigma}\left(-k^2 \pm \frac{e^2}{4\pi^2}\mu_A k\right)\mathbf{B} \rightarrow \text{instability @ } k < \frac{e^2}{4\pi^2}\mu_A$$

CME is the one in equilibrium & hydrodynamic approx.

→ How we can see the effect in non-eq. system?

⇒ We often use kinetic/Boltzman eqn. in any field of physics (cosmology, astrophysics, nuclear, cond.mat. ...)

distribution function $f(t, \mathbf{x}(t), \mathbf{p}(t))$

$$\frac{D}{Dt}f = \frac{\partial f}{\partial t} + \dot{\mathbf{x}}\frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{p}}\frac{\partial f}{\partial \mathbf{p}} = 0 \qquad \text{(collisionless)}$$

↖ current non-conservation is not seen.

for just $\dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{p}} = \mathbf{v}\times\mathbf{B} \quad \left(\mathbf{v} = \frac{\partial \varepsilon}{\partial \mathbf{p}}\right)$

⇒ How shall we modify the kinetic equation?

\# not just add the collision term!

2. Derivation of the chiral kinetic equation.

$$\frac{\partial f}{\partial t} + \underline{\underline{\dot{x}}} \frac{\partial f}{\partial x} + \underline{\underline{\dot{p}}} \frac{\partial f}{\partial p} = C[f]$$

receives modification.

Hamiltonian of massless charged Weyl particle

$$H = \vec{\sigma}\cdot(\vec{p} - e\vec{A}) + eA^0$$

Action

$$S = \int dt \ \vec{p}\cdot\dot{\vec{x}} - \vec{\sigma}\cdot\vec{p} + \dot{\vec{x}}\cdot\vec{A} - A^0 \qquad (e=1)$$

Quantum correction

$$S_{eff} = \int dt \ \vec{p}\cdot\dot{\vec{x}} + \vec{A}\cdot\dot{\vec{x}} - A^0 - |\vec{p}| - a_p\cdot\dot{p}$$

a_p satisfies $\Omega \equiv \nabla_p \times a_p = \dfrac{p}{2|\vec{p}|^3}$

$\hookrightarrow$ Berry connection $\longleftrightarrow$ Berry curvature

$\Rightarrow$ EOM

$$\sqrt{G}\,\dot{x} = \frac{p}{|p|} + E\times\Omega + B\left(\frac{p}{|p|}\cdot\Omega\right)$$

$$\sqrt{G}\,\dot{p} = E + \frac{p}{|p|}\times B + \Omega\cdot(E\cdot B)$$

$$G = (1+\Omega B)^2$$

$$\rightarrow \sqrt{G}\frac{\partial f}{\partial t} + \left(\frac{p}{|p|} + E\times\Omega + B\left(\frac{p}{p}\cdot\Omega\right)\right)\frac{\partial f}{\partial x} + \left(E + \frac{p}{|p|}\times B + \Omega\cdot(E\cdot B)\right)\frac{\partial f}{\partial p} = C[f]$$

a_p : $U(1)$ gauge field in the momentum space

Ω_p : magnetic field $\Rightarrow$ "Berry monopole" with Berry curvature, invariant measure of phase space reads

$$\frac{\sqrt{G}}{(2\pi)^3}\, d^3x\, d^3p$$

$$\because\quad \frac{\partial}{\partial t}\sqrt{G} + \frac{\partial}{\partial \mathbf{x}}\left(\sqrt{G}\,\dot{\mathbf{x}}\right) + \frac{\partial}{\partial \mathbf{p}}\left(\sqrt{G}\,\dot{\mathbf{p}}\right) = 2\pi\, \mathbf{E}\cdot\mathbf{B}\, \delta^{(3)}(\mathbf{p})$$

$\leftarrow \nabla_{\mathbf{p}}\cdot\boldsymbol{\Omega} \leftarrow$ source from Berry monopole

$$\rightarrow\quad n(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{G}\, f$$

$$\mathbf{j}(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{G}\, \dot{\mathbf{x}}\, f$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{|\mathbf{p}|} f + \underbrace{\mathbf{E}\times\int \frac{d^3p}{(2\pi)^3}\boldsymbol{\Omega} f}_{\text{anomalous Hall effect}} + \underbrace{\mathbf{B}\cdot\int \frac{d^3p}{(2\pi)^3}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\Omega}\right) f}_{\text{chiral magnetic effect}}$$

$$\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\Omega} = \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\frac{\mathbf{p}}{2|\mathbf{p}|^3} = \frac{1}{2|\mathbf{p}|^2}$$

$$\mathbf{B}\cdot\int_0^\infty \frac{dp}{4\pi^2}\cdot f \qquad \text{for } f = \begin{cases} 1 & (E<\mu) \\ 0 & (E>\mu) \end{cases} \quad \begin{matrix}\text{zero temp.} \\ \text{Fermi-distribution}\end{matrix}$$

$$= \frac{\mu}{4\pi^2}\mathbf{B}$$

last term of $\dot{\mathbf{p}}$

$$\frac{\partial n}{\partial t} + \nabla\cdot\mathbf{j} = \frac{1}{4\pi^2}\mathbf{E}\cdot\mathbf{B}\, f(\mathbf{p}=0)$$

$\Rightarrow$ Anomaly equation is reconstructed!

from the contribution @ $\mathbf{p}=0$

Detailed calculations

☒ S_{eff}

$$\langle f | e^{iH(t_f-t_i)} | i \rangle = \left[\int \mathcal{D}x \, \mathcal{D}p \; P\exp\left[i \int_{t_i}^{t_f} (\mathbf{p}\cdot\dot{\mathbf{x}} - \sigma\cdot\mathbf{p})\, dt\right]\right]_{fi}$$

$$\exp\left[-i\int_{t_i}^{t_f} \sigma\cdot\mathbf{p}\, dt\right] \to \exp[-i\sigma\cdot\mathbf{p}_n\Delta t]\times \exp[-i\sigma\cdot\mathbf{p}_{n-1}\Delta t]\times\cdots\times\exp[-i\sigma\cdot\mathbf{p}_1\Delta t]$$

introduce a unitary matrix $V_{\mathbf{p}}$ so that $\qquad \mathbf{p}_n = \mathbf{p}_f$

$$V_{\mathbf{p}}^\dagger \, \sigma\cdot\mathbf{p} \, V_{\mathbf{p}} = |\mathbf{p}|\sigma_3 \qquad \mathbf{p}_1 = \mathbf{p}_i$$

$$\to V_{\mathbf{p}_n}\exp[-i|\mathbf{p}_n|\sigma_3\Delta t]\, V_{\mathbf{p}_n}^\dagger V_{\mathbf{p}_{n-1}} \exp[-i|\mathbf{p}_{n-1}|\sigma_3\Delta t]\, V^\dagger_{\mathbf{p}_{n-1}} V_{\mathbf{p}_{n-2}}$$
$$\times\cdots\times \exp[-i|\mathbf{p}_1|\sigma_3\Delta t]\, V^\dagger_{\mathbf{p}_1}$$

$$\begin{aligned} V_{\mathbf{p}_2}^\dagger V_{\mathbf{p}_1} &= V_{\mathbf{p}_2}^\dagger \left(V_{\mathbf{p}_2} + \Delta\mathbf{p}\cdot\nabla_{\mathbf{p}} V_{\mathbf{p}_2}\right) \\ &= \mathbb{1} + \Delta\mathbf{p}\cdot V^\dagger_{\mathbf{p}_2}\nabla_{p} V_{\mathbf{p}_2} \\ &\simeq \exp[-i\hat{a}_{\mathbf{p}}\cdot\Delta\mathbf{p}] \qquad \hat{a}_{\mathbf{p}} = iV^\dagger_{\mathbf{p}}\nabla_{\mathbf{p}} V_{\mathbf{p}} \end{aligned}$$

$$\to V_{\mathbf{p}_n}\exp[-i|\mathbf{p}_n|\sigma_3\Delta t]\times\exp[-i\hat{a}_{\mathbf{p}_n}\cdot\dot{\mathbf{p}}_n\Delta t]$$
$$\times\exp[-i|\mathbf{p}_{n-1}|\sigma_3\Delta t]\times\exp[-i\hat{a}_{\mathbf{p}_{n-1}}\cdot\dot{\mathbf{p}}_{n-1}\Delta t]$$
$$\times\cdots$$
$$\times\exp[-i|\mathbf{p}_1|\sigma_3\Delta t]\times\exp[-i\hat{a}_{\mathbf{p}_1}\cdot\dot{\mathbf{p}}_1\Delta t]\, V^\dagger_{\mathbf{p}_1}$$

$$\Rightarrow \langle f|e^{iH(t_f-t_i)}|i\rangle = \Big[V_{\mathbf{p}_f}\int\mathcal{D}x\,\mathcal{D}p\; P\exp\left[i\int_{t_i}^{t_f}(\mathbf{p}\cdot\dot{\mathbf{x}} - |\mathbf{p}|\sigma_3 - \hat{a}_{\mathbf{p}}\cdot\dot{\mathbf{p}})\,dt\right] \times V^\dagger_{\mathbf{p}_i}\Big]$$

"Classical regime"

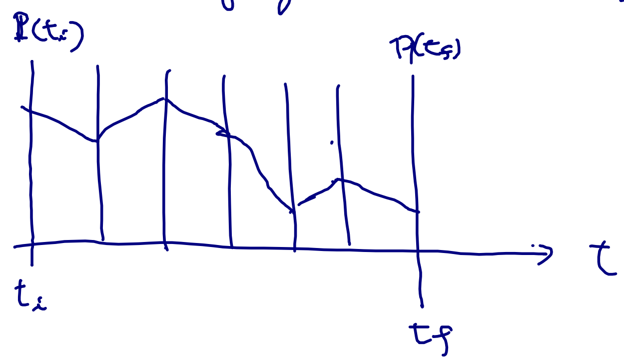
→ omit the off-diagonal components

⇒ + helicity mode

→ $S_{eff} = \int_{t_i}^{t_f} (\mathbf{p}\cdot\dot{x} - |\mathbf{p}| - a_p \cdot \dot{\mathbf{p}})\,dt$

$a_{\mathbf{p}} \equiv [\hat{a}_p]_{11}$

▨ $\hat{a}_{\mathbf{p}}$ is the gauge field under SU(2) spin



each $\mathbf{P}(t_j)$ we have the degree of freedom of $\vec{\sigma}$ basis $U(1)$

we take the basis $\sigma\cdot\mathbf{p} = |\mathbf{p}|\,\sigma_3$

↳ rotate with $U_{\mathbf{p}} \in U(2)$

→ $-|\mathbf{p}|\sigma_3 \Rightarrow -|\mathbf{p}|\,U_p^{\dagger}\sigma_3 U_{\mathbf{p}}$

$\hat{a}_p \to U_{\mathbf{p}}^{\dagger}\,\hat{a}_{\mathbf{p}} U_{\mathbf{p}} + i U_{\mathbf{p}}^{\dagger}\nabla_{\mathbf{p}} U_{\mathbf{p}}$

↳ this is the transportation law of U(2) gauge field

$a_{\mathbf{p}}$ → U(1) gauge field

Explicitly we write

$$V_{\mathbb{P}} = \begin{pmatrix} P_1 - iP_2 & P_3 - P \\ -P_3 + P & P_1 + iP_2 \end{pmatrix} \times \frac{1}{\sqrt{2P(P-P_3)}}$$

we obtain

$$\Omega = \nabla_{\mathbb{P}} \times a_P = \frac{\mathbb{P}}{2|p|^3}$$

singular @ $|\mathbb{P}| = 0$

$\leftrightarrow$ $|\mathbb{P}| = 0$ $\to$ 2 helicity modes are degenerate.

▨ EOM

$$S_{eff} = \int dt\, \mathbb{P}\cdot\dot{x} + \mathbb{A}\cdot\dot{x} - \underbrace{|\mathbb{P}| - A^0}_{= H} - a\cdot\dot{\mathbb{P}}$$

$$= \int dt \left[-\omega_a \dot{\xi}^a - H(\xi) \right]$$

$$\begin{pmatrix} \omega_x = -(\mathbb{P} + \mathbb{A}(x)) \\ \omega_{\mathbb{P}} = a \end{pmatrix} \to \omega_{ab} \dot{\xi}^b = -\partial_a H$$

$$\omega_{ab} \equiv \partial_a \omega_b - \partial_b \omega_a$$

$$\left(\frac{\partial}{\partial \xi^a} (\omega_b \dot{\xi}^b) = \frac{\partial \omega_b}{\partial \xi^a} \dot{\xi}^b + \omega_b \frac{\partial}{\partial \xi^a} \dot{\xi}^b \right)$$

$$\to \dot{\xi}^a = \{H, \xi^a\} = -\underbrace{\{\xi^a, \xi^b\}}_{= \omega_{ab}^{-1}} \partial_b H$$

$$\omega_{x_i x_j} = \partial_{x_i} \omega_{x_j} - \partial_{x_j} \omega_{x_i} = -F_{ij} = -\varepsilon_{ijk} B_k$$

$$\omega_{p_i p_j} = \partial_{p_i} \omega_{p_j} - \partial_{p_j} \omega_{p_i} = \varepsilon_{ijk} \Omega_k$$

$$\omega_{x_i p_j} = \partial_{x_i} \omega_{p_j} - \partial_{p_j} \omega_{x_i} = \delta_{ij}$$

$$\rightarrow \{p_i, p_j\} = -\frac{\varepsilon_{ijk} B_k}{1+B\cdot\Omega}$$

$$\{x_i, x_j\} = \frac{\varepsilon_{ijk}\Omega_k}{1+B\cdot\Omega}$$

$$H = |p| + A^0(x)$$

$$\{p_i, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1+B\Omega}$$

$$\rightarrow \dot{x}_i = -\{x^i, x^j\}\frac{\partial H}{\partial x_j} - \{x^i, p_j\}\frac{\partial H}{\partial p_j}$$

$$\boxed{\partial_t A^i = 0}$$

$$= -\frac{\varepsilon_{ijk}\Omega_k}{1+B\cdot\Omega}\cdot\frac{\partial A^0}{\partial x_j} + \frac{\delta_{ij} + \Omega_i B_j}{1+B\cdot\Omega}\,\frac{p_i}{p}$$

$$\sqrt{G}\,\dot{x}_i = \frac{p_i}{p} + \varepsilon_{ijk} E_j \Omega_k + B_i\,(\Omega\cdot p/|p|)$$

$$\dot{p}_i = -\{p^i, x^j\}\frac{\partial H}{\partial x_j} - \{p^i, p^j\}\frac{\partial H}{\partial p_j}$$

$$= -\frac{\delta_{ij} + \Omega_i B_j}{1+B\cdot\Omega}\cdot\frac{\partial A^0}{\partial x_j} + \frac{\varepsilon_{ijk} B_k}{1+B\cdot\Omega}\cdot\frac{p_j}{p}$$

$$\sqrt{G}\;\dot{p}_i = E_i + \Omega_i(E\cdot B) + (\hat{p}\times B)_i$$

OK!

3. Chiral magnetic wave / an application

right-handed Weyl fermion with BG $\mu = c$

with $\mathbb{E} = 0$. but $\mathbb{B} \neq 0$

$$f = f_0 - \frac{\partial f_0}{\partial \varepsilon} h_\pm (t, x, p), \quad f_0 = \frac{1}{e^{\beta\varepsilon}+1}$$

$\pm$: particle/antiparticle & $\mathbb{B} \to -\mathbb{B}$, $\Omega \to -\Omega$

linearlized kinetic equation

$$h_\pm = \int e^{-i\omega t + kx} h_\pm \, d^3x dt$$

$$\to -i\omega h_\pm + \dot{x}\left(ik \pm \mathbb{B} \times \frac{\partial}{\partial p}\right) h_\pm = I_\pm [h_+, h_-]$$

$$C[f] = -\frac{\partial f^0}{\partial \varepsilon} I_\pm [h_+, h_-]$$

(Ω-term does not contribute to this order)

$$h \equiv h_+ - h_-$$

$$\to -i\omega \langle h \rangle + ik \cdot \langle \dot{x} h \rangle = 0$$

for small $\underset{=0}{k}$, $h =$ const. is a solution $\to h = 1 + O(k)$

$$\to \omega = k \cdot \langle \dot{x} \rangle + O(k^2)$$

$$= k \cdot \left\langle \frac{1}{\sqrt{G}} (v \cdot \Omega) \mathbb{B} \right\rangle$$

$$\langle \ \rangle = -\frac{2}{\chi} \int \frac{d^3p}{(2\pi)^3} \sqrt{G} \frac{\partial f^0(\varepsilon)}{\partial \varepsilon} \cdots$$

$$\to \omega = \frac{\mathbb{B}}{4\pi\chi} \cdot k \qquad -\frac{\chi}{2} = \int \frac{d^3p}{(2\pi)^3} \sqrt{G} \frac{\partial f^0}{\partial \varepsilon} = \frac{T^2}{6} + O\left(\frac{\mathbb{B}^2}{T^2}\right)$$

$\Rightarrow$ wave with group velocity

$$v = \frac{B}{4\pi\chi}$$

"chiral magnetic wave"