Time complexification for particle production

1. Introduction

* Why are particles produced?

EoM for mode function of free massive scalar field in Minkowski space is

$$\ddot{\phi}_k + \underbrace{(k^2+m^2)}_{\substack{|||\\ \omega_k^2}} \phi_k = 0 .$$

ω_k^2 effective frequency

Then, states is in a Hilbert space spanned by these two bases;

positive freq. mode $v \propto e^{-i\omega_k t}$

negative 〃 $v^* \propto e^{+i\omega_k t}$

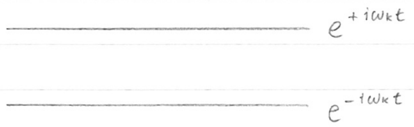
$v^* \hat{a} + v \hat{a}^\dagger$

negative frequency mode content rate

||

particle number

If ω_k is constant,

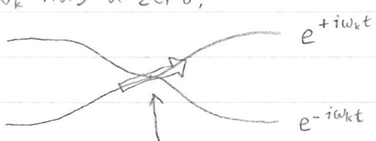


Two modes never intersect

$\Rightarrow$ ratio never changes

$\Rightarrow$ No particle production

If ω_k has a zero,

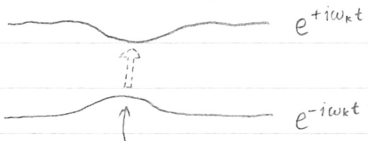


some fraction flows into another mode at a zero

$\Rightarrow$ particle production

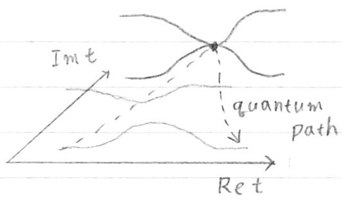
Zero plays crucial roles in particle production!

However, in most cases, ω_k varies but has no zero.



transition is classically forbidden, but quantum mechanically possible!

How to explain this?



Even if ω_k has no zero on real t axis,

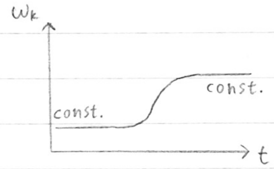
it has zeros in COMPLEX t plane.

Quantum path picks up these zeros

$\Rightarrow$ We have to complexify time variable in order to calculate particle production correctly!

2. Bogoliubov coefficients

First, let's briefly review how to calculate particle number from mode function.
Assume that ω_k is asymptotically constant at infinite past & future.



$\downarrow$

We can uniquely define positive & negative frequency modes

Hereafter, f_k and g_k denote a positive frequency mode in infinite past and future, respectively.

If an initial state is a vacuum state,

state		vacuum (positive freq. mode)
f_k		f_k
$\downarrow$ evolves according to EoM		$\uparrow$
$\tilde{f}_k = \alpha_k g_k + \underbrace{\beta_k g_k^*}$	$\Longleftarrow$	$g_k \leftarrow$ defined by diagonalization of Hamiltonian

Bogoliubov transformation

This gives particle number density as $n_k = |\beta_k|^2$

g_k and β_k can be easily derived as $e^{-i\Omega_k t}/\sqrt{2\Omega_k}$, where $\Omega_k = \lim_{t\to+\infty}\omega_k$, and $-iW[\tilde{f}_k, g_k]$, respectively.

The main problem is "how to calculate (or approximate) $\tilde{f}_k$."

* WKB approximation (example of a method 'without' time complexification)

Substituting the ansatz

$$\tilde{f}_k = \frac{1}{\sqrt{2W_k(t)}}\exp\left[i\int_{t_0}^{t} W_k(t')\,dt'\right]$$

into EoM, we find that W_k must satisfy

$$W_k^2 = \omega_k^2 \underbrace{- \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right]}.$$

$\dot{\omega}_k/\omega_k \ll \omega_k$

If ω_k varies slowly (almost adiabatically), this term is perturbation

Then, this equation can be regarded as a recurrence relation w.r.t. the time differential order

$$\downarrow$$

$$W_k^{(n+2)\,2} = \omega_k^2 - \frac{1}{2}\left[\frac{W_k^{(n)\prime\prime}}{W_k^{(n)}} - \frac{3}{2}\left(\frac{W_k^{(n)\prime}}{W_k^{(n)}}\right)^2\right], \quad \text{where } W_k^{(0)} = \omega_k .$$

But this is a divergent expansion, and then a higher order is NOT necessarily more accurate.
It requires time complexification to obtain a better approximation!

3. Lefschetz-thimble and Stokes phenomenon

3-1. Picard-Lefschetz theory   1403.1277, 1510.03435

This is originally a theory about contour of path integral, but it is useful also for particle production.
This theory tells us how to take an appropriate contour for calculating the d-dimensional integral

$$\int_{-\infty}^{\infty} d^d x\, e^{-S(x)} \qquad\qquad d^d x = dx^1 \cdots dx^d$$

$$\Big\downarrow \text{ complexify the variables } x^j \to z^j = x^j + i y^j$$

$$\int_D d^d z\, e^{-S(z)}$$

D: d-dim. submanifold   if $d=1$, this is a contour

We have to choose a domain D such that the integral converges.

$\Downarrow$ in other word

$$\mathrm{Re}[S(z)] \to \infty \qquad (|z^i| \to \infty \text{ in } D)$$

☆ How to make such a domain D?

Start from a critical point z_a such that

$$\left.\frac{\partial S}{\partial z}\right|_{z=z_a} = 0 . \qquad \text{if } d=1, \text{ this is a saddle point}$$

We want to increase the real part of $S(z)$ from here.
$\mathrm{Re}[S(z)]$ monotonically increases along the gradient flow $z(\tau)$ defined as

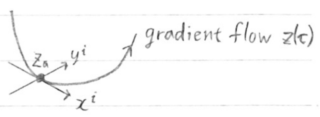
$$\frac{dz}{d\tau} = \overline{\frac{\partial S}{\partial z}} . \qquad \text{but we can't start } z(\tau) \text{ from } z = z_a \text{ since } \frac{\partial S}{\partial z} = 0 \text{ there} \ldots$$

$\overline{\partial S/\partial z}$: complex conjugate of $\partial S/\partial z$

Exercise

> Prove that $\mathrm{Re}[S(z)]$ monotonically increases and $\mathrm{Im}[S(z)]$ is constant along $z(\tau)$.
> Hint: use Cauchy-Riemann equations.

We can choose a coordinate z such that
$x^1 = \mathrm{Re}(z^1)$ direction is along the gradient flow around $z = z_a$.
Then, $\mathrm{Re}[S(z)]$ is expanded around $z = z_a$ as

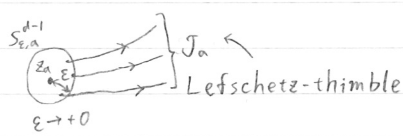


$$\mathrm{Re}[S(z)] = \mathrm{Re}[S(z_a)] + \underbrace{[(x^1-x_a^1)^2 + \cdots + (x^d-x_a^d)^2]}_{\text{increase}} - [(y^1-y_a^1)^2 + \cdots + (y^d-y_a^d)^2] + \mathcal{O}[(z-z_a)^3]$$

increase $\because$ x^i is along $z(\tau)$

Then, we can start gradient flows from a point on (d-1)-dim. sphere such that

$$S^{d-1}_{\varepsilon,a} : (x^1-x_a^1)^2 + \cdots + (x^d-x_a^d)^2 = \varepsilon^2$$



$\Rightarrow \mathcal{J}_a$: set of gradient flows starting from a point on $\lim_{\varepsilon\to 0} S^{d-1}_{\varepsilon,a}$

$\mathcal{J}_a$: This is Lefschetz-thimbles!

But all Lefschetz-thimbles don't necessarily contribute to the original integral.
How to determine which Lefschetz-thimbles are relevant?

A 'downward' gradient flow $\tilde{z}(\tau)$ such that

$$\frac{d\tilde{z}}{d\tau} = -\overline{\frac{\partial S(\tilde{z})}{\partial \tilde{z}}}$$

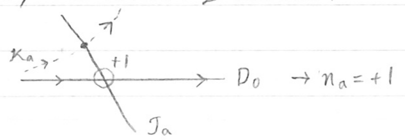
opposite sign

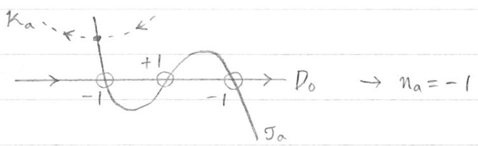
is very useful for criteria. We define a 'conjugation' of Lefschetz-thimble $\mathcal{J}_a$ as

$\mathcal{K}_a$: set of downward gradient flows
starting from a point on an infinitesimal sphere $(y^1-y^1_a)^2+\cdots+(y^d-y^d_a)^2=\varepsilon^2$ $(\varepsilon\to+0)$

Then, we can reconstruct the integral domain by the following criteria;

Let n_a be the intersection number between $\mathcal{J}_a$ and the original domain D_0 counting by the following manner,





with the same direction as $\mathcal{K}_a \rightarrow +1$ with the direction opposite to $\mathcal{K}_a \rightarrow -1$

Then, the integral domain is reconstruct as $D_0 \rightarrow \sum_a n_a \mathcal{J}_a$

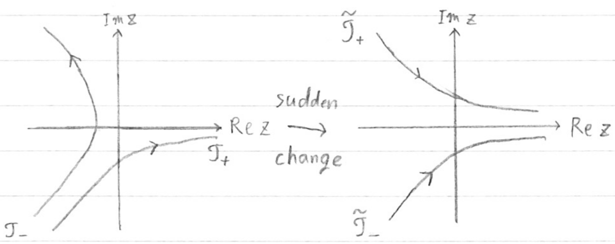
This construction method for integral domain D is called as Picard-Lefschetz theory.

3-2. Stokes phenomenon 0911.4692, 1001.2933

In a certain situation, the structure of Lefschetz-thimbles suddenly changes as the argument of $S(z)$ varies.
For example, consider the Airy function

$$Ai(\lambda)=\frac{1}{2\pi i}\int dz\, e^{\frac{z^3}{3}-\lambda z} \quad\rightarrow S(z)=-\frac{z^3}{3}+\lambda z \quad \text{critical points: } z=0, \pm\sqrt{3\lambda}$$

When λ changes from $+0$ to -0,
the structure of Lefschetz-thimbles
suddenly changes as shown in the right figure.
This is the Stokes phenomenon.



$\lambda > 0$ $\lambda < 0$

Since Lefschetz-thimbles correspond to the integral domain, their deformation means the change of asymptotic forms.
Actually, when $x\to+\infty$,

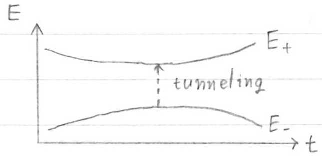
$$Ai(x)\sim\frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}, \quad Ai(-x)\sim\frac{\sin\left(\frac{2}{3}x^{3/2}+\frac{1}{4}\pi\right)}{\sqrt{\pi}\,x^{1/4}}$$

In our case, "the change of asymptotic forms" means the change of vacuum state, and hence, particle production!

4. Lefschetz-thimble inspired approach 1907.12224

Let's consider a two-level problems with two adiabatic states $|\phi_\pm(t)\rangle$ such that

$$\underset{\substack{\uparrow\\ \text{Hamiltonian}}}{\hat{H}(t)}|\phi_\pm(t)\rangle = \underset{\substack{\uparrow\\ \text{adiabatic energy } E_- < E_+}}{E_\pm(t)}|\phi_\pm(t)\rangle .$$



A physical state $|\psi(t)\rangle$ is expanded by $|\phi_\pm(t)\rangle$ as

$$|\psi(t)\rangle = \sum_{j=\pm} c_j(t)\, e^{-i\mathcal{E}_j(t)}|\phi_j(t)\rangle , \quad \text{where } \mathcal{E}_j(t) = \int_{t_0}^{t} dt'\, E_j(t') .$$

$\downarrow$ substitute into the Schrödinger equation

$$\dot{c}_\pm(t) = \pm\, \xi(t)\, e^{\pm i\Delta(t)} c_\mp(t) , \quad \text{where } \xi(t) = \langle \phi_-(t)|\tfrac{d}{dt}|\phi_+(t)\rangle ,\ \Delta(t) = \int_{t_0}^{t} dt' [E_+(t) - E_-(t)] .$$

The transition probability from $|\phi_-(-\infty)\rangle$ to $|\phi_+(+\infty)\rangle$ (= produced particle number) is derived as

$$c_+(+\infty) = \int_{-\infty}^{\infty} dt\, \xi(t)\, e^{i\Delta(t)} c_-(t) \simeq \int_{-\infty}^{\infty} dt\, e^{-F(t)} \quad \text{with } c_-(-\infty) = 1,\ c_+(-\infty) = 0 .$$

We can apply Picard-Lefschetz theory!

Assuming that $c_-(t) \approx 1$ ($\Leftrightarrow c_+(t) \ll 1$), we can take

$$F(t) = -i\Delta(t) + \ln \xi(t) .$$

if we also neglect $\xi(t)$, it is DDP method (Dykhne 1962, Davis & Pechukas 1976)

Around each critical point t_a, $F(t)$ is expanded as

$$F(t) = F(t_a) + \frac{|F''(t_a)|}{2} r^2 e^{i(\phi_a + 2\theta)} + \mathcal{O}(r^3) , \quad \text{where } \phi_a = \arg(F''(t_a)),\ r = t - t_a .$$

$\downarrow$ Gaussian integrate along $\mathcal{J}_a$ ($\phi_a + 2\theta = 0$)

$$\boxed{c_+(+\infty) \simeq \sum_a n_a\, e^{i\theta_a - F(t_a)} \sqrt{\frac{2\pi}{|F''(t_a)|}}}$$

θ_a: interference among Lefschetz-thimbles

5. Stokes phenomenon inspired approach 1004.2509, 1405.0302

Since the WKB approximation is a divergent expansion, there is $\underbrace{\text{an optimal order}}$ to be truncated.

We can find this from the Stokes phenomenon!

Berry 1990 showed that truncating the WKB approximation at the optimal order, we obtain

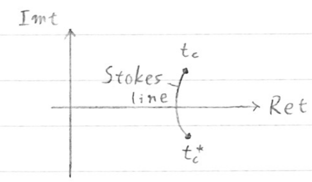
$$\beta_k(t) \approx \frac{i}{2}\,\mathrm{Erfc}(-\sigma_k(t))\, e^{-F_k^{(0)}},$$

for the detailed derivation, see §A.2 in 1903.08842

where

singulant $F_k(t) := 2i\int_{t_c}^{t} dt'\,\omega_k(t')$ t_c ⋯ critical point in the upper half-plane

natural time evolution parameter $\sigma_k(t) := \dfrac{\mathrm{Im}[F_k(t)]}{\sqrt{2\,\mathrm{Re}[F_k(t)]}}$

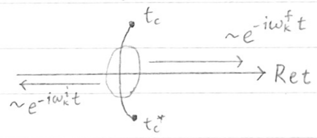


amplitude $F_k^{(0)} := i\int_{t_c}^{t_c^*} dt\,\omega_k(t)$.

along Stokes line

border on which Stokes phenomenon occurs

this means



mode function has different asymptotic forms on each side of Stokes line.

⇓

particle production mainly occurs around Stokes lines!

If there are several critical points $t_1, t_2, \cdots$, then resultant particle number is

$$\boxed{\begin{aligned} n_k &\approx \left|\sum_a e^{2i\theta_{k,a}} e^{-F_{k,a}^{(0)}}\right|^2, \\ \text{where}\quad \theta_{k,a} &:= \int_{\mathrm{Re}\,t_1}^{\mathrm{Re}\,t_a} dt\,\omega_k(t) \\ F_{k,a}^{(0)} &:= i\int_{t_a}^{t_a^*} dt\,\omega_k(t) \end{aligned}}$$

← interference among Stokes lines

← particle production around a-th Stokes line

t_a is a-th critical point

Acknowledgement

Takuya Shimazaki gave me his great master thesis. It really helped me to understand Picard-Lefschetz theory.