MHD in a Cylindrical Shearing Box

Takeru K. Suzuki

School of Arts & Science, U. Tokyo

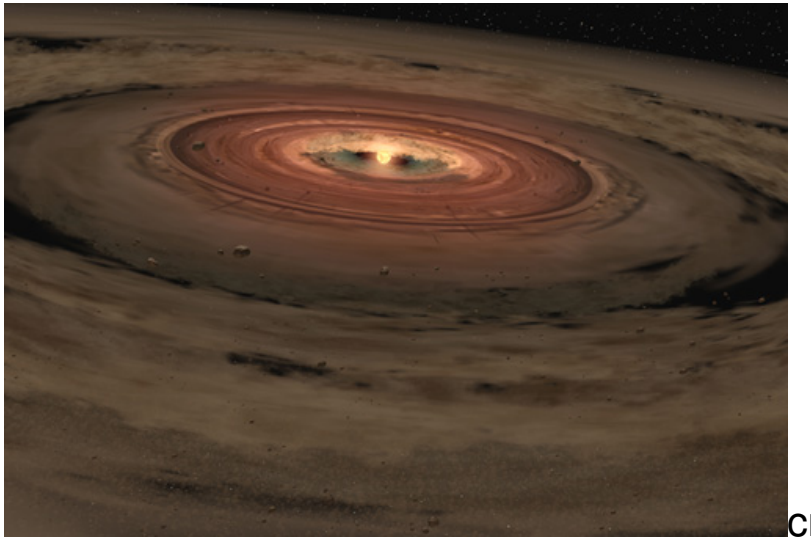
September 6th, 2018

Thanks to XC40@YITP & ATERUI@CfCA/NaOJ

Accretion Disks

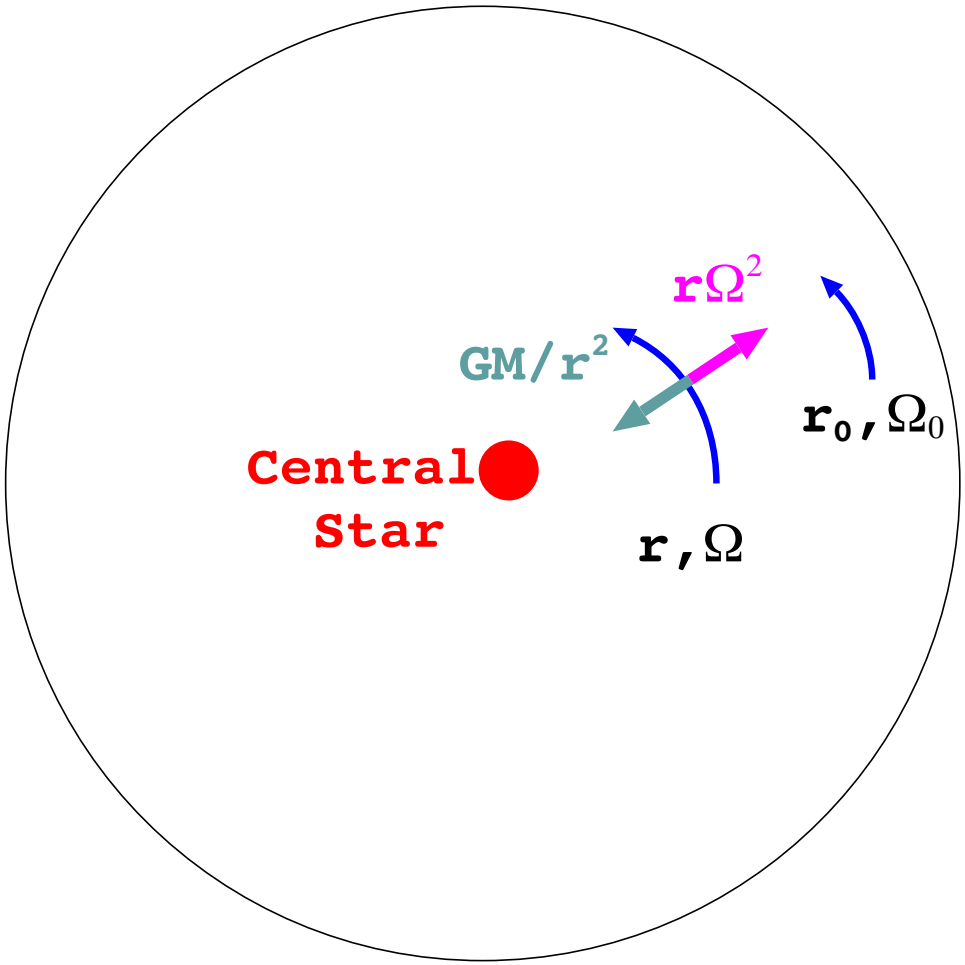
Some Examples

- Accretion Disks around Black Holes
- Active Galactic Nuclei (SMBH)
- Galactic Disks
- Protoplanetary Disks

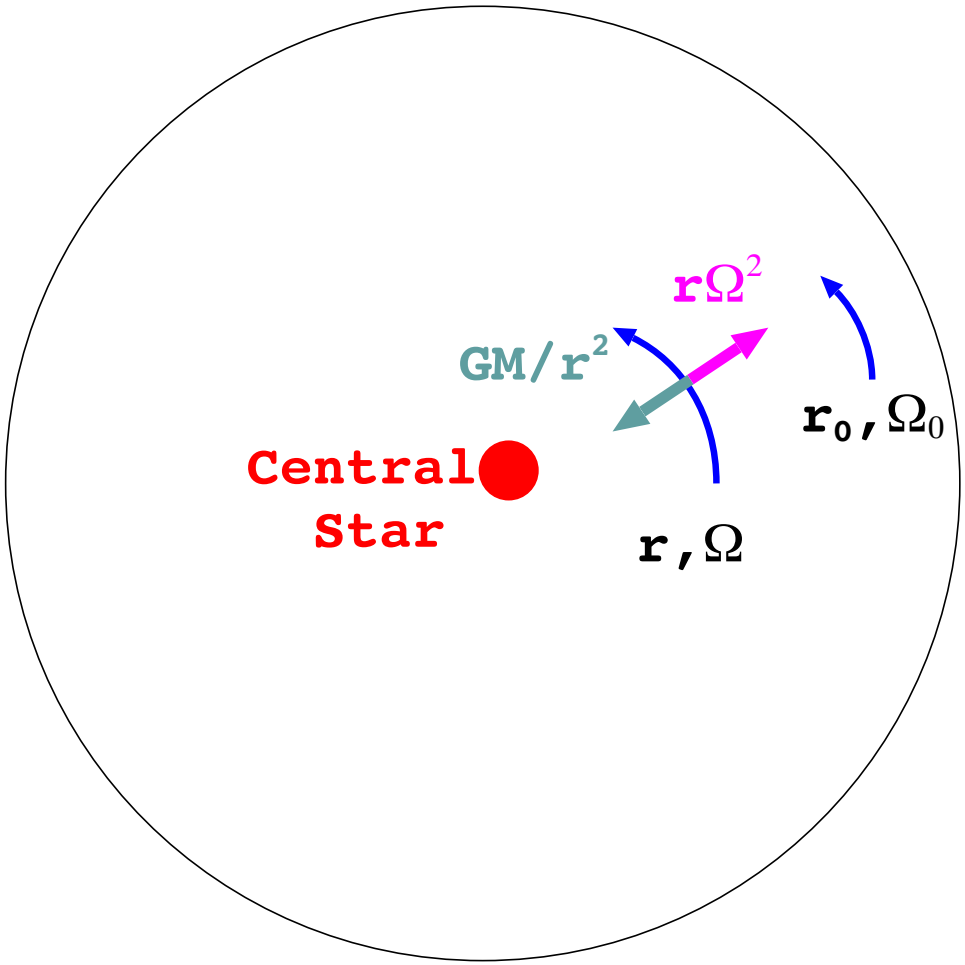


credit: NASA

Ang. Mom. Transport & Mass Accretion

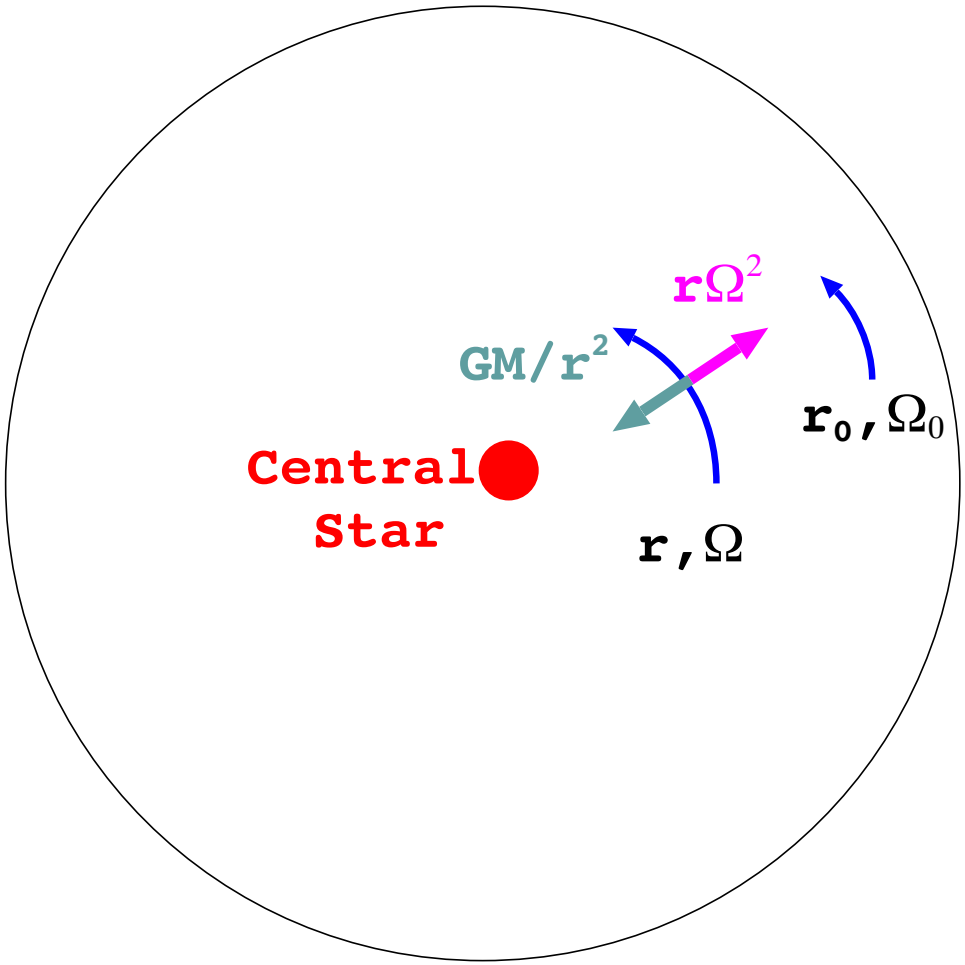


Ang. Mom. Transport & Mass Accretion



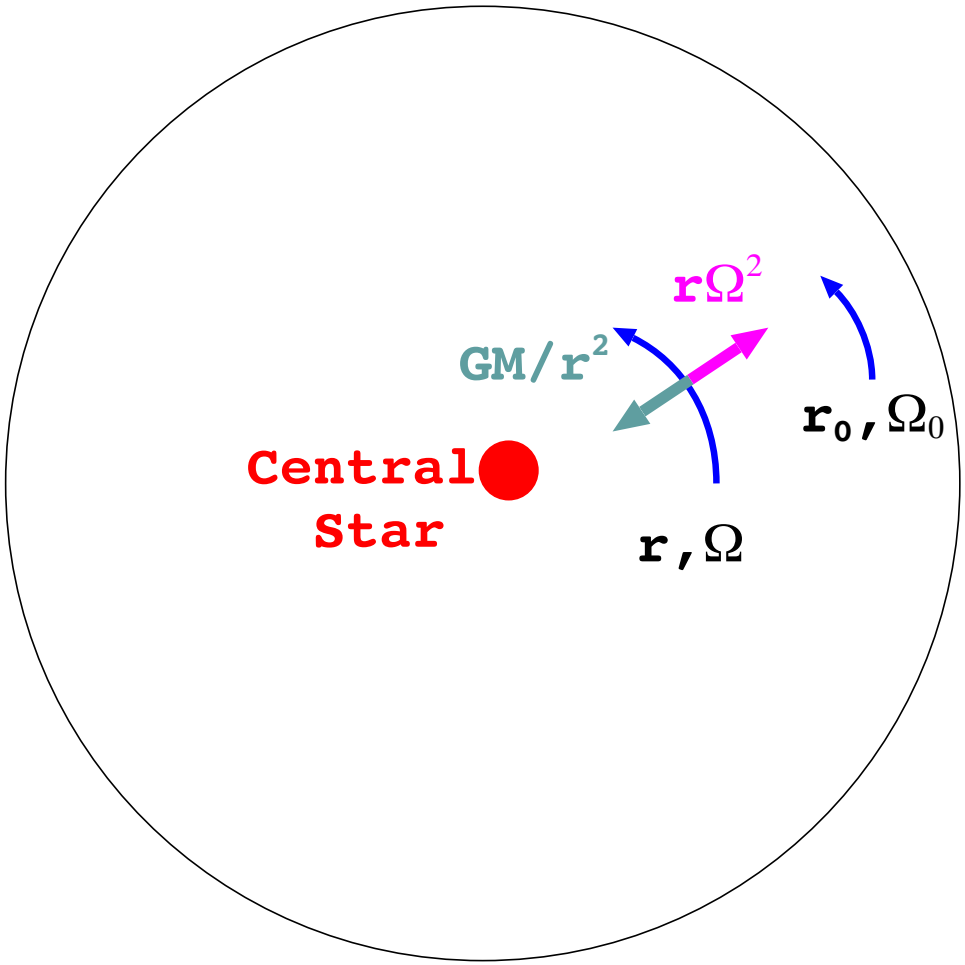
- If Ang. Mom. conserved:
  $\Omega = \Omega_0 (\frac{r_0}{r})^2$

Ang. Mom. Transport & Mass Accretion



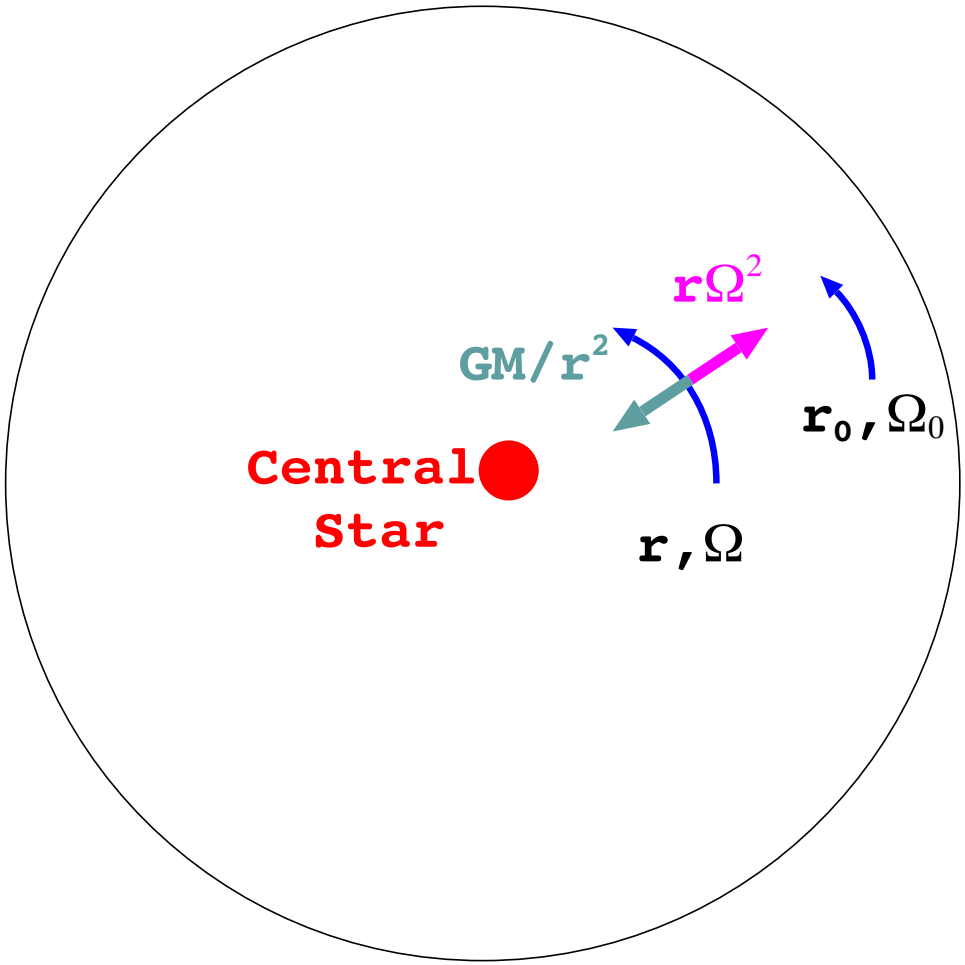
- If Ang. Mom. conserved:
  $\boldsymbol{\Omega = \Omega_0 (\frac{r_0}{r})^2}$
- Centrifugal force at $\boldsymbol{r}$:
  $\boldsymbol{r\Omega^2 = r_0 \Omega_0^2 (\frac{r_0}{r})^3}$

Ang. Mom. Transport & Mass Accretion



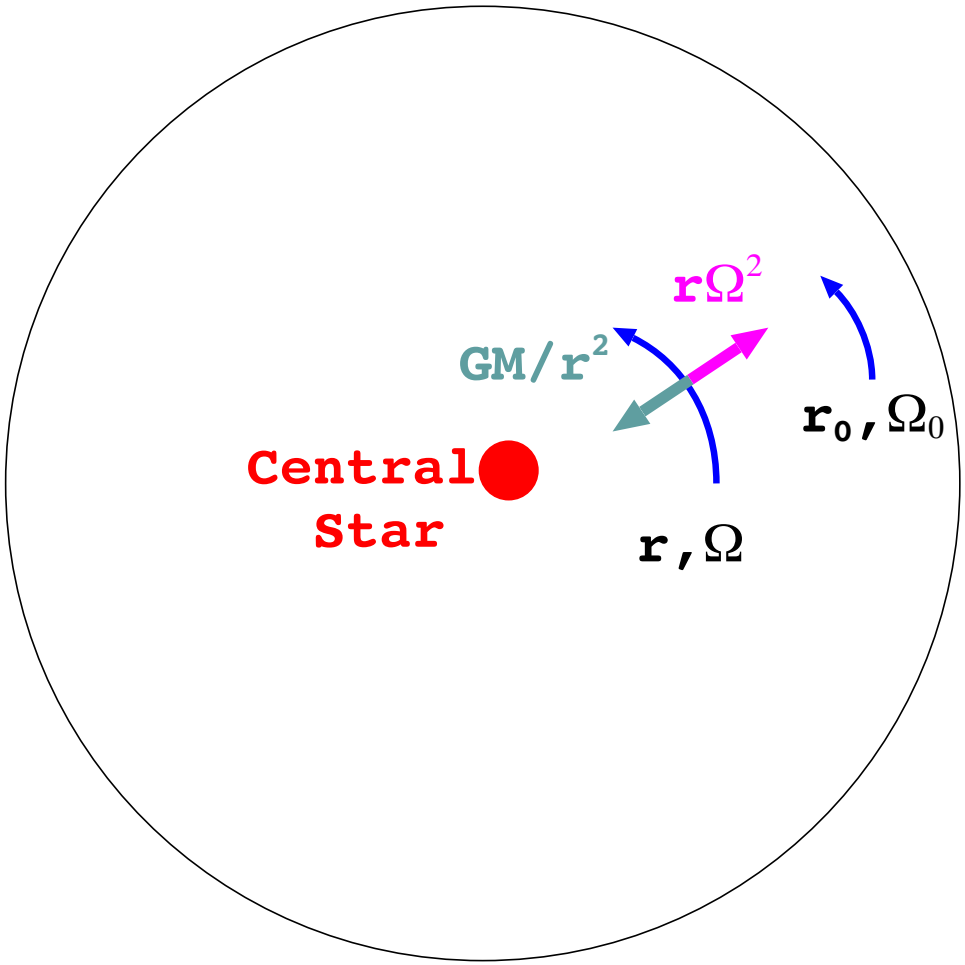
- If Ang. Mom. conserved:
  $\Omega = \Omega_0(\frac{r_0}{r})^2$
- Centrifugal force at r :
  $r\Omega^2 = r_0\Omega_0^2(\frac{r_0}{r})^3$
- Gravity at r :
  $\frac{GM}{r_0^2}(\frac{r_0}{r})^2$

Ang. Mom. Transport & Mass Accretion



- If Ang. Mom. conserved: $\Omega = \Omega_0(\frac{r_0}{r})^2$
- Centrifugal force at r : $r\Omega^2 = r_0\Omega_0^2(\frac{r_0}{r})^3$
- Gravity at r : $\frac{GM}{r_0^2}(\frac{r_0}{r})^2$
- If $r_0\Omega_0^2 = GM/r_0^2$, $\Rightarrow r\Omega^2 > GM/r^2$

Ang. Mom. Transport & Mass Accretion



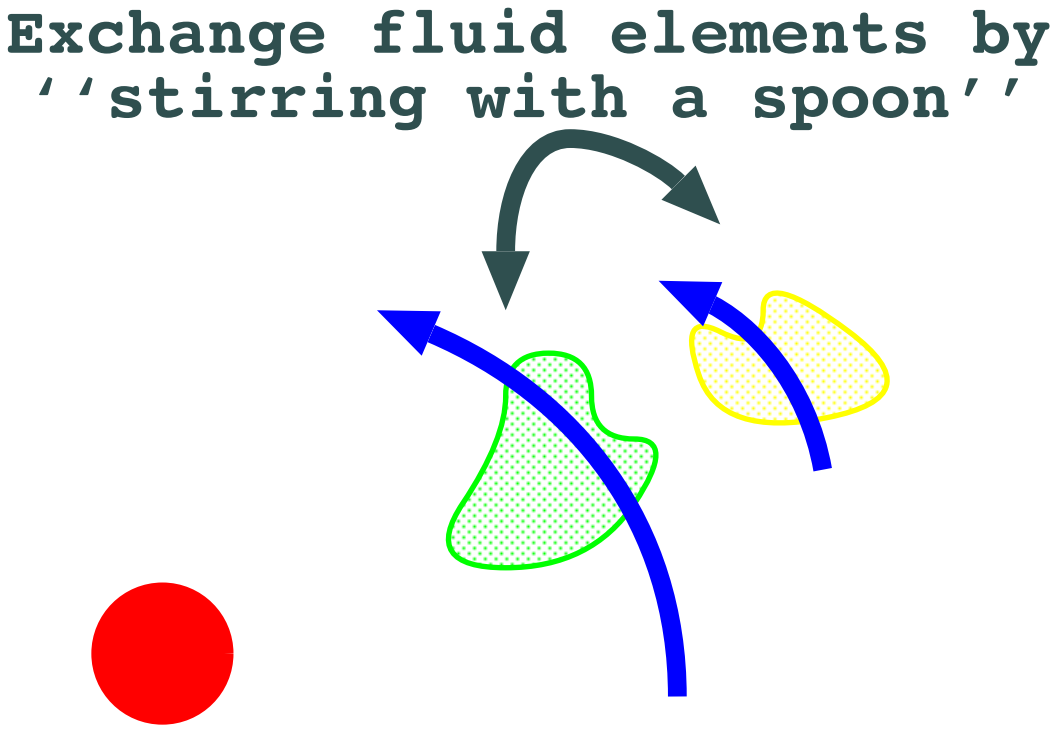
- If Ang. Mom. conserved:
  $\Omega = \Omega_0(\frac{r_0}{r})^2$
- Centrifugal force at r :
  $r\Omega^2 = r_0\Omega_0^2(\frac{r_0}{r})^3$
- Gravity at r :
  $\frac{GM}{r_0^2}(\frac{r_0}{r})^2$
- If $r_0\Omega_0^2 = GM/r_0^2$,
  $\Rightarrow r\Omega^2 > GM/r^2$
- Mass does NOT accrete.

(Rayleigh Criterion)

Turbulence in Accretion Disks

Turbulence ⇒ Macroscopic (effective) Viscosity

- Outward Transport of Angular Momentum
- Inward Accretion of Matters



MHD in an Accretion Disk

Suzuki & Inutsuka 2014

Magneto-Rotational Instability (MRI) –linear analyses–

Balbus & Hawley 1991

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\frac{dv}{dt} + \frac{1}{\rho}\nabla(p + \frac{B^2}{8\pi}) - \frac{(B\cdot\nabla)B}{4\pi\rho} + \nabla\Phi = 0$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B - \eta \nabla \times B)$$

$$\rho\frac{de}{dt} = -p\nabla \cdot v + \frac{\eta}{4\pi}|\nabla \times B|^2$$

- axisymmetric perturbation: $\propto \exp(-i\omega t + ik_r r + ik_z z)$
- Gravity by a central star $\nabla\Phi \approx (\frac{GM}{r^2}, 0, \frac{GMz}{r^3})$
- Assuming $B_0 = (0, 0, B_{z,0})$, ideal MHD ($\eta = 0$), & incompressive ($k_r \delta v_r + k_z \delta v_z = 0$)

Dispersion relation :

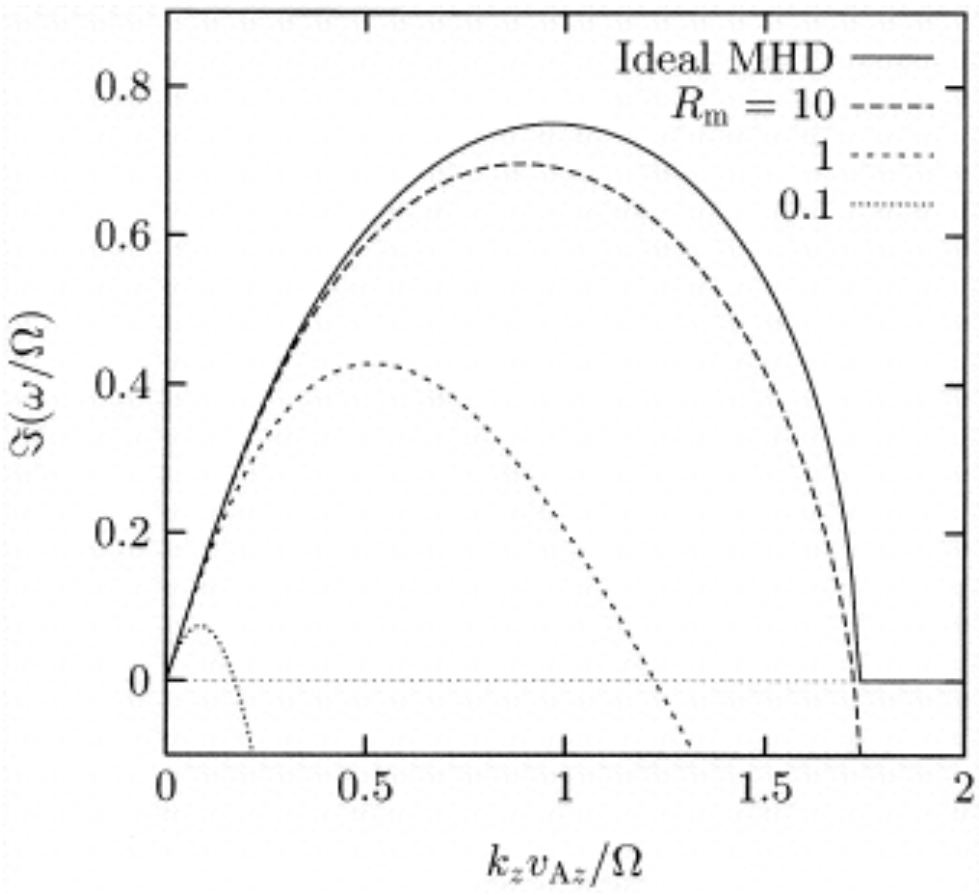
$$\omega^4 - (2v_{A,z}^2 k_z^2 + \kappa^2 \frac{k_z^2}{k^2})\omega^2 + v_{A,z}^4 k_z^4 + (\kappa^2 - 4\Omega^2)v_{A,z}^2 \frac{k_z^4}{k^2} = 0$$

where κ : epicycle frequency (= Ω for Kepler rotation)

$$v_{A,z} = B_{z,0} / \sqrt{4\pi\rho}$$

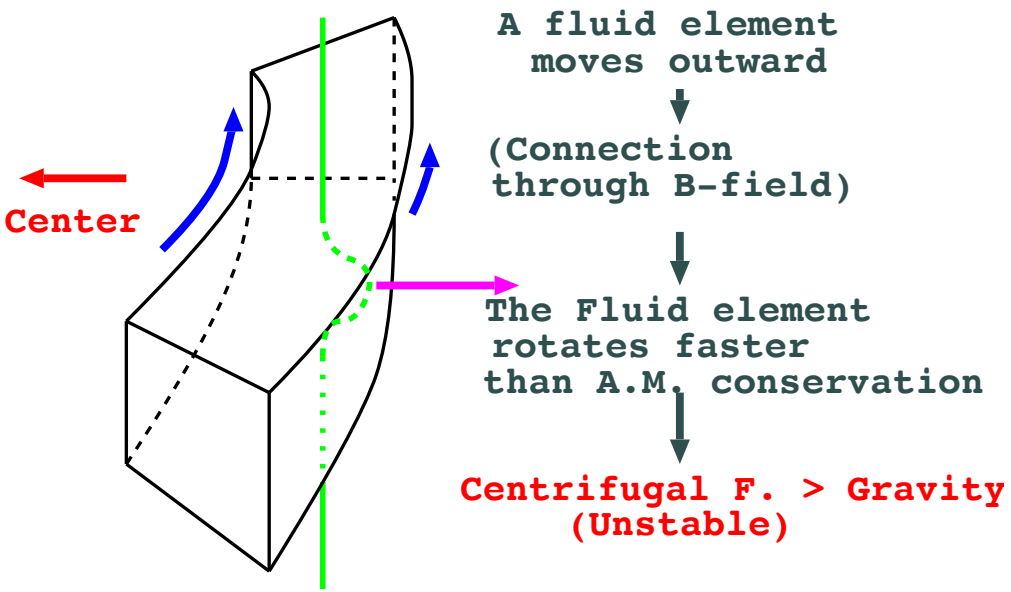
MRI –Dispersion Relation–

Sano & Miyama 1999



- Always unstable for the weak $\boldsymbol{B}$ ($\beta = \frac{8\pi p}{B^2} \gtrsim 1$)
- The growth rate $\sim \Omega^{-1}$

Magneto-Rotational Instability (MRI)



Unstable under

- Weak B-fields
- (inner-fast) Differential Rotation

MHD in Cartesian Shearing Box (CaSB)

Hawley, Gammie, & Balbus 1995

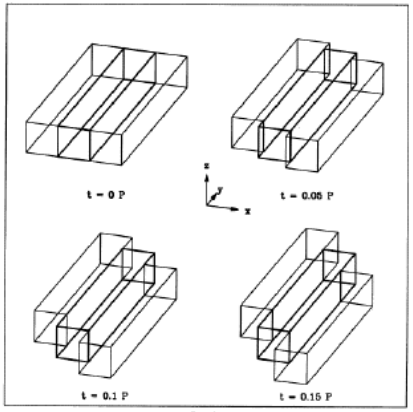


FIG. 1a

MHD in Cartesian Shearing Box (CaSB)

- Local Cartesian coordinate with co-rotating with $\mathbf{\Omega_0}$.
  (neglect curvature)

Hawley, Gammie, & Balbus 1995

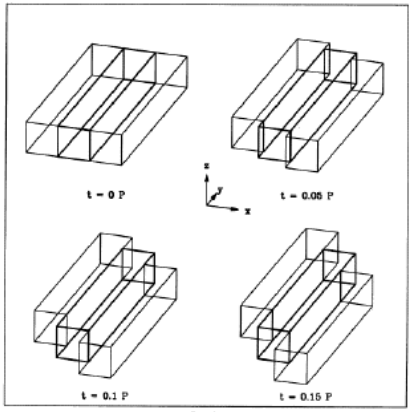


FIG. 1a

MHD in Cartesian Shearing Box (CaSB)

- Local Cartesian coordinate with co-rotating with $\mathbf{\Omega_0}$. (neglect curvature)
- $\boldsymbol{x = r - r_0}$; $\boldsymbol{y \leftrightarrow \phi}$-direction

Hawley, Gammie, & Balbus 1995

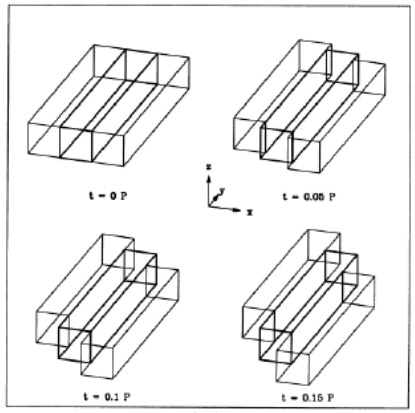


FIG. 1a

MHD in Cartesian Shearing Box (CaSB)

- Local Cartesian coordinate with co-rotating with Ω_0. (neglect curvature)
- $x = r - r_0$; $y \leftrightarrow \phi$-direction
- Basic equations for Keplerian rotation ($\Omega_0 = \sqrt{GM/r^3}$)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\nabla_x\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_x}{4\pi\rho} + 2\Omega_0 v_y + 3\Omega_0^2 x$$

$$\frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\nabla_y\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_y}{4\pi\rho} - 2\Omega_0 v_x$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\nabla_z\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_z}{4\pi\rho} - \Omega_0^2 z$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B - \eta \nabla \times B)$$

$$\nabla \cdot B = 0$$

Hawley, Gammie, & Balbus 1995

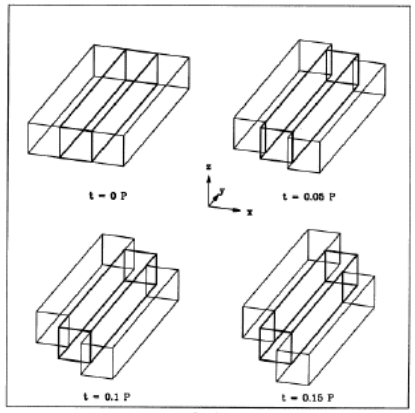


Fig. 1a

MHD in Cartesian Shearing Box (CaSB)

- Local Cartesian coordinate with co-rotating with Ω_0. (neglect curvature)
- $x = r - r_0$; $y \leftrightarrow \phi$-direction
- Basic equations for Keplerian rotation ($\Omega_0 = \sqrt{GM/r^3}$)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\nabla_x\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_x}{4\pi\rho} + 2\Omega_0 v_y + 3\Omega_0^2 x$$

$$\frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\nabla_y\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_y}{4\pi\rho} - 2\Omega_0 v_x$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\nabla_z\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_z}{4\pi\rho} - \Omega_0^2 z$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B - \eta \nabla \times B)$$

$$\nabla \cdot B = 0$$

- An Isothermal Equation of State

Hawley, Gammie, & Balbus 1995

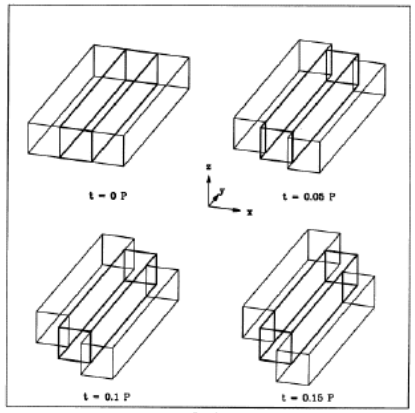


Fig. 1a

MHD in Cartesian Shearing Box (CaSB)

- Local Cartesian coordinate with co-rotating with Ω_0. (neglect curvature)
- $x = r - r_0$; $y \leftrightarrow \phi$-direction
- Basic equations for Keplerian rotation ($\Omega_0 = \sqrt{GM/r^3}$)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\nabla_x\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_x}{4\pi\rho} + 2\Omega_0 v_y + 3\Omega_0^2 x$$

$$\frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\nabla_y\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_y}{4\pi\rho} - 2\Omega_0 v_x$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\nabla_z\left(p + \frac{B^2}{8\pi}\right) + \frac{(B\cdot\nabla)B_z}{4\pi\rho} - \Omega_0^2 z$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B - \eta \nabla \times B)$$

$$\nabla \cdot B = 0$$

Hawley, Gammie, & Balbus 1995

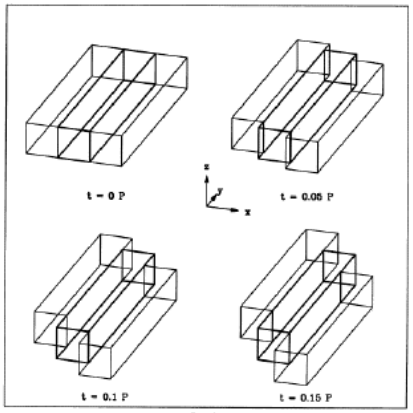


Fig. 1a

- An Isothermal Equation of State
- Steady-state solution
  - $B = (0, B_y, B_z)$ & $v = (0, -\frac{3}{2}\Omega_0 x, 0)$
  - $\rho = \rho_0 \exp(-z^2/H^2)$ ($H^2 \equiv 2c_s^2/\Omega_0^2$): hydrostatic equilibrium

Cartesian Shearing Box Simulations

Hawley et al. 1995; Matsumoto & Tajima 1995; ...

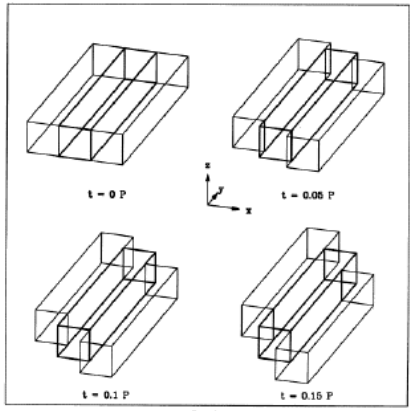
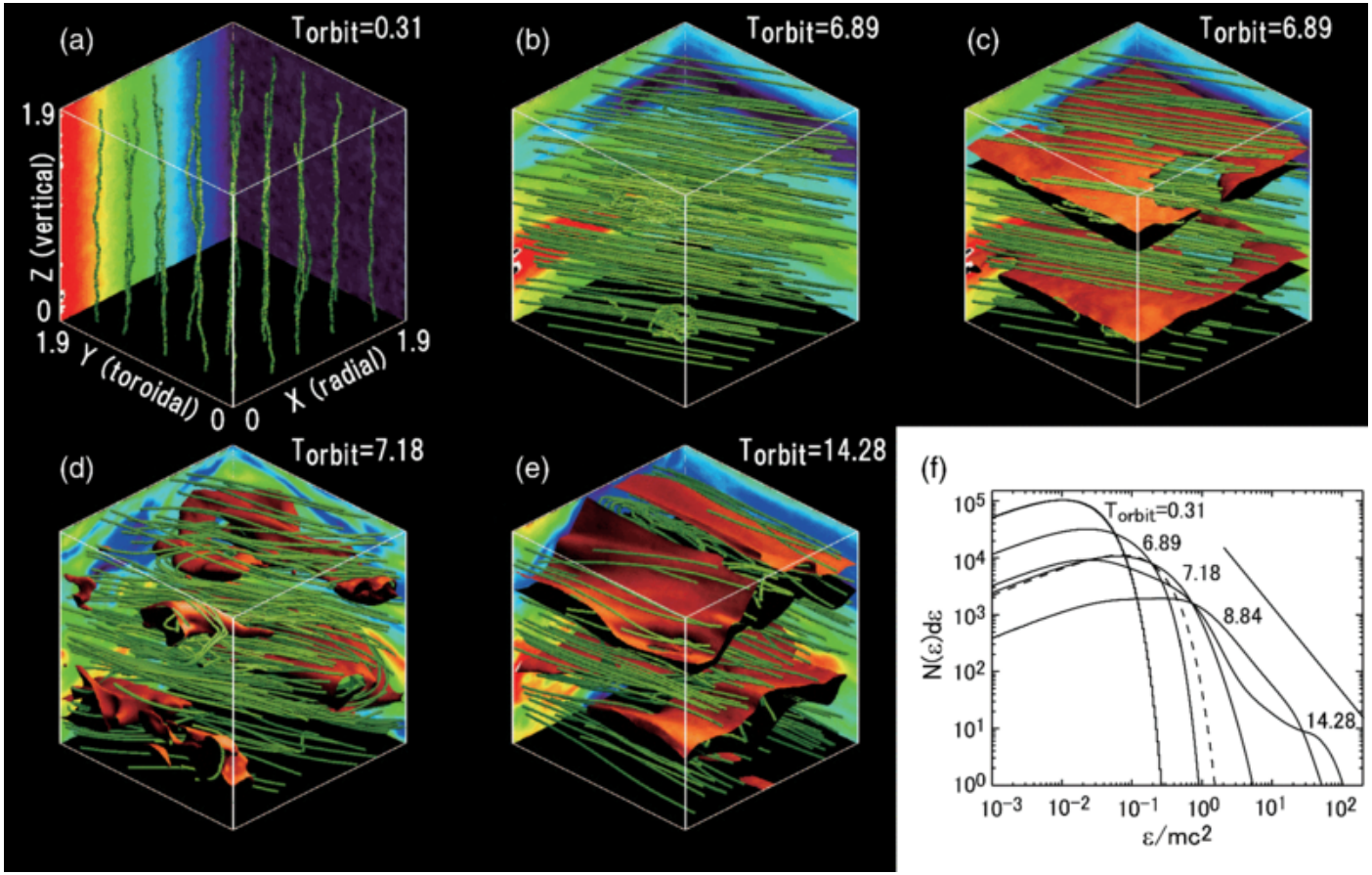


FIG. 1a

Suzuki & Inutsuka 2009

Applications of CaSB

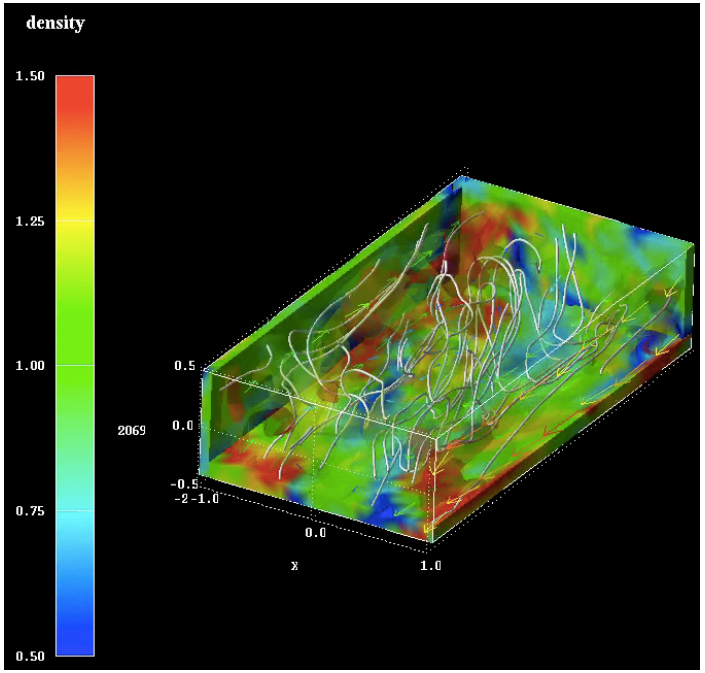
- PIC simulation in CaSB



Hoshino 2013; 2015; Shirakawa & Hoshino 2014

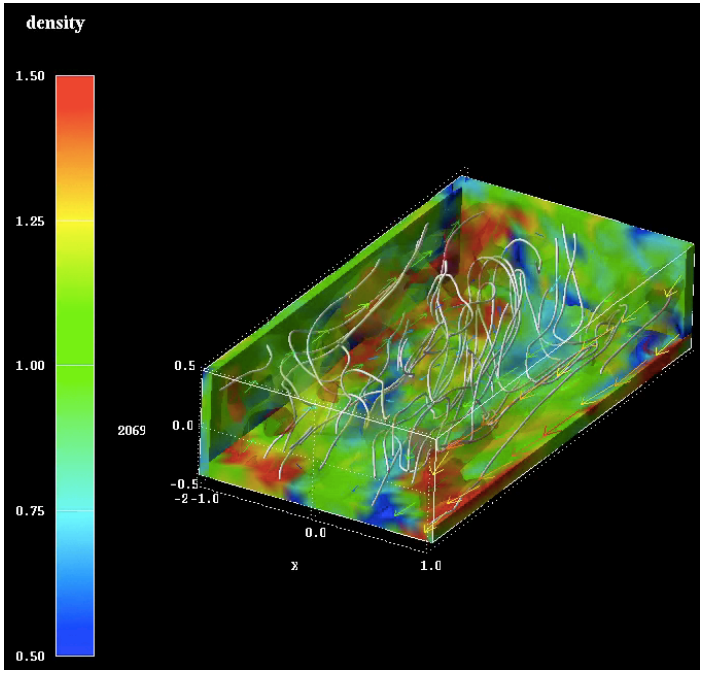
- MHD + non-thermal particles Kimura+ 2016

Some Disadvantages of CaSB



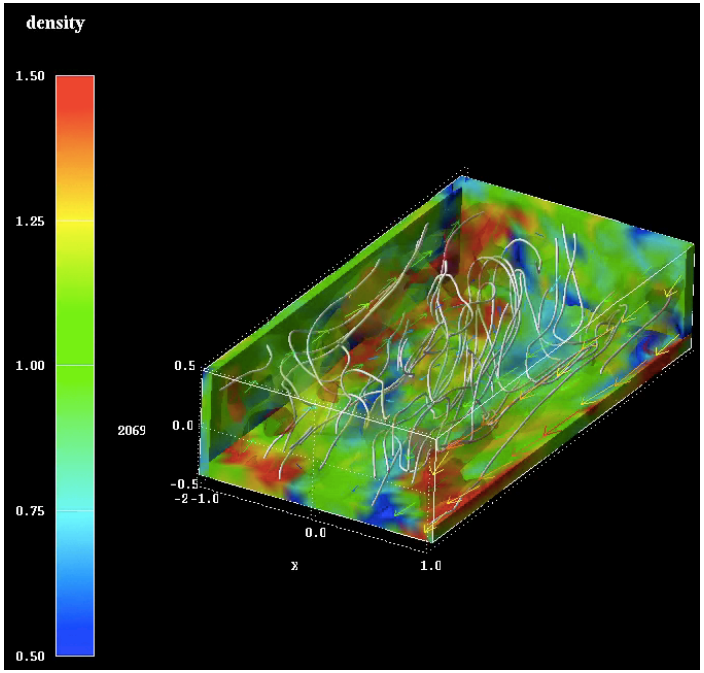
Some Disadvantages of CaSB

- Neglect the Curvature



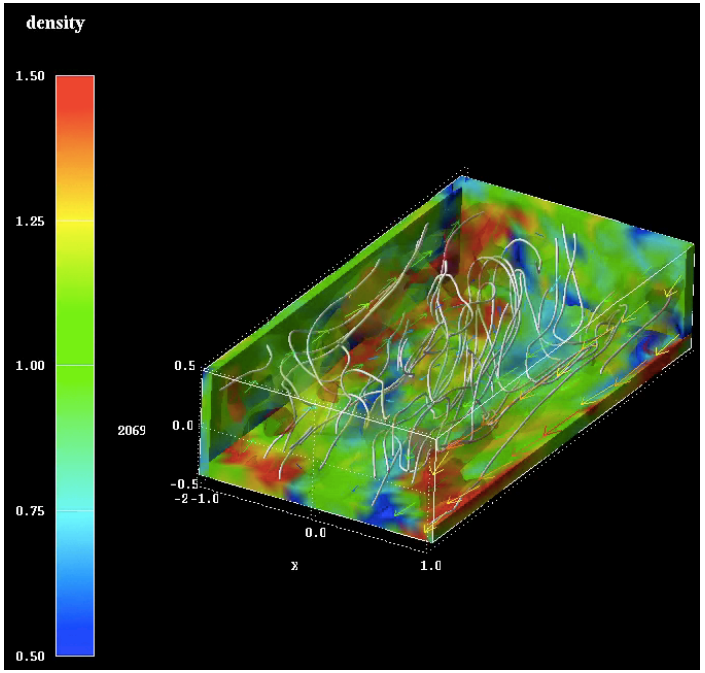
Some Disadvantages of CaSB

- Neglect the Curvature
- Symmetry to the $\pm x$ direction
  The central star can be located on either left or right side



Some Disadvantages of CaSB

- Neglect the Curvature
- Symmetry to the $\pm x$ direction
  The central star can be located on either left or right side
- No Gas Accretion



A New Aproach

A New Aproach

- Break the Symmetry

A New Aproach

- Break the Symmetry
- Introduce the Curvature

A New Aproach

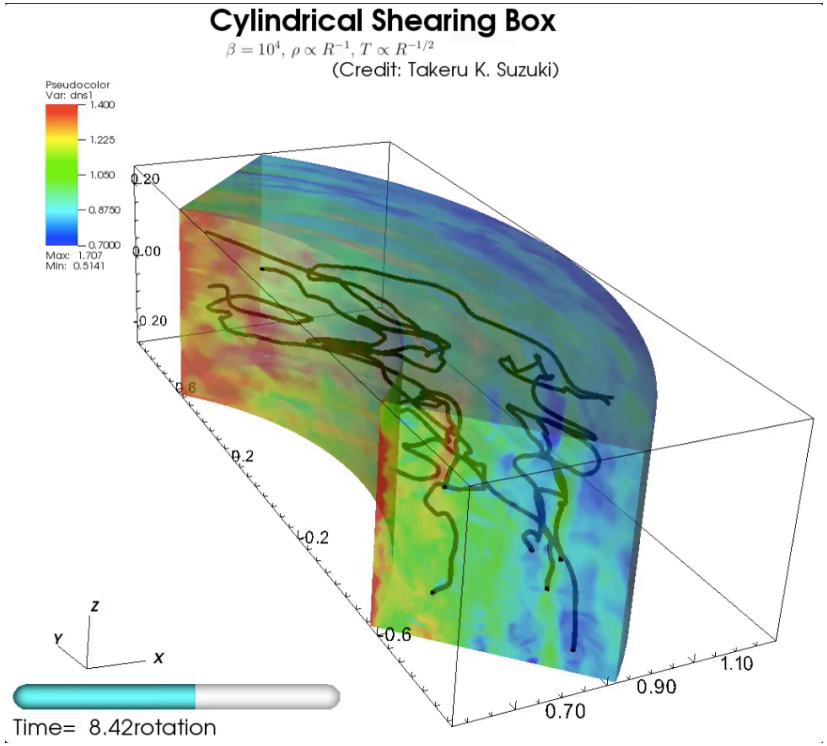
- Break the Symmetry
- Introduce the Curvature
  $\Rightarrow$ can handle the net accretion ?

A New Aproach

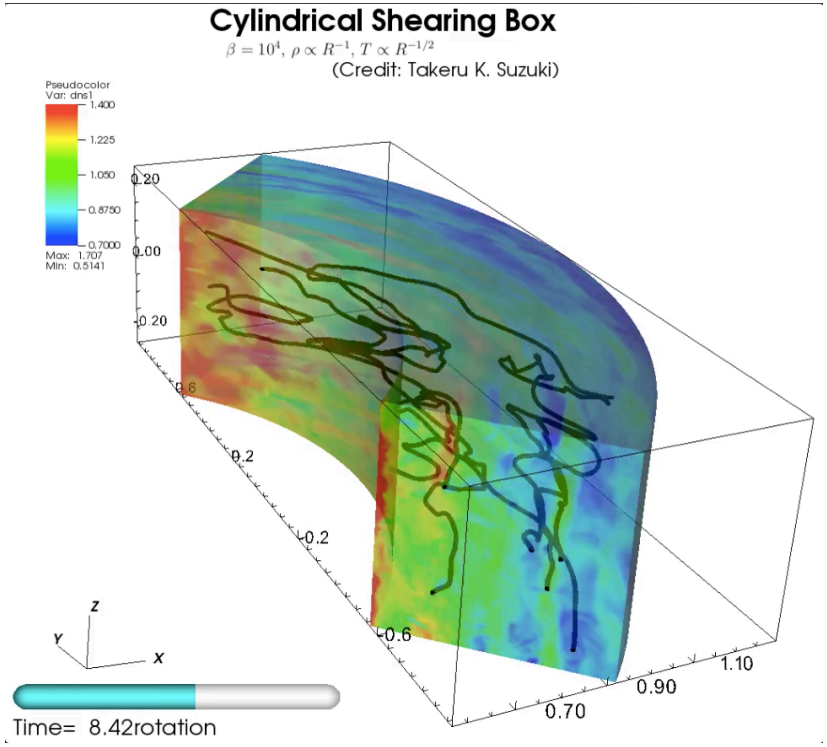
- Break the Symmetry
- Introduce the Curvature

  ⇒ can handle the net accretion ?

⇒ Let's try "Cylindrical Shearing Box (CySB)"

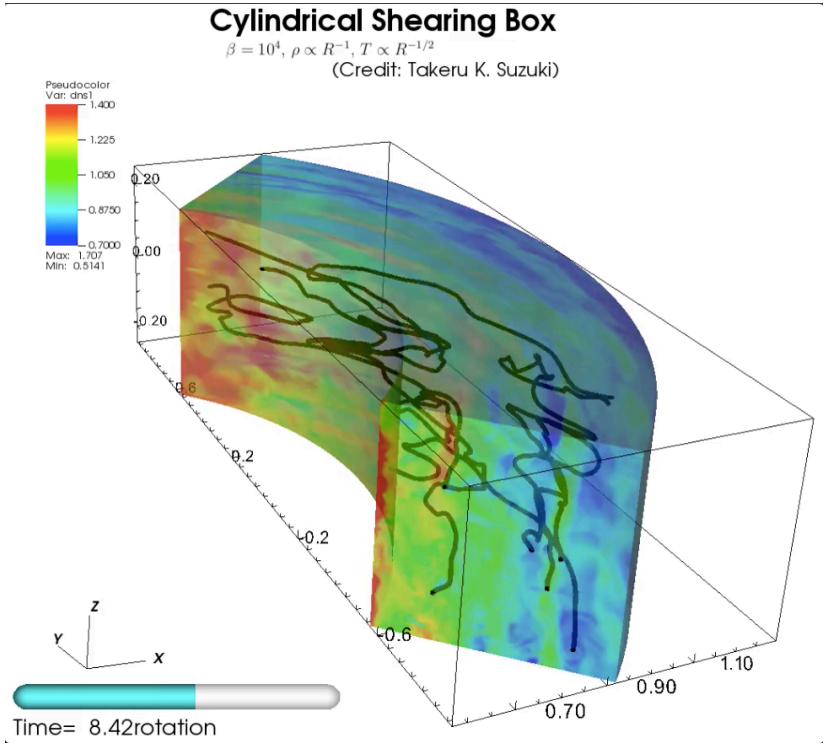


Cylindrical Shearing Box (CySB)



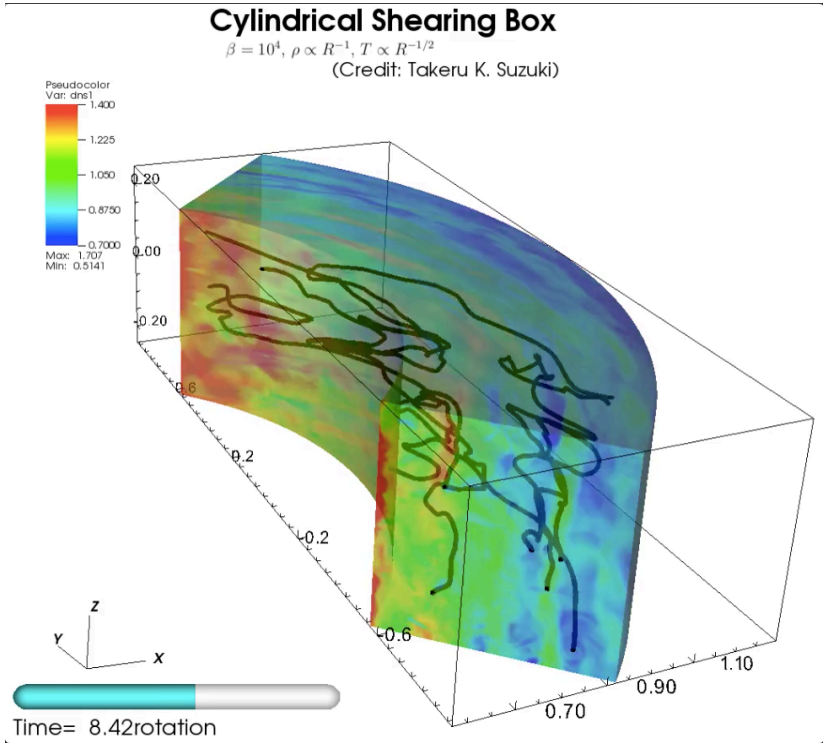
Cylindrical Shearing Box (CySB)

Key : Boundary Condition at $R_{\pm}$



Cylindrical Shearing Box (CySB)

Key : Boundary Condition at $\boldsymbol{R}_{\pm}$



Radial Boundary Condition

$\Leftarrow$ Conservation Laws

of Mass+Momentum+(Energy)+$\boldsymbol{B}$

Equations

Equations

- Mass:

$$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$

Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$

Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$
- Momentum– ϕ (Angular Momentum):
  $$\partial_t(\rho v_\phi R) + \partial_R[(\rho v_R v_\phi + B_R B_\phi/4\pi)R^2] + \cdots = 0$$

Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$
- Momentum– ϕ (Angular Momentum):
  $$\partial_t(\rho v_\phi R) + \partial_R[(\rho v_R v_\phi + B_R B_\phi/4\pi)R^2] + \cdots = 0$$
- Momentum– z:
  $$\partial_t(\rho v_z) + R^{-1}\partial_R(\rho v_R v_z R) + \cdots = 0$$

Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$
- Momentum– ϕ (Angular Momentum):
  $$\partial_t(\rho v_\phi R) + \partial_R[(\rho v_R v_\phi + B_R B_\phi/4\pi)R^2] + \cdots = 0$$
- Momentum– z:
  $$\partial_t(\rho v_z) + R^{-1}\partial_R(\rho v_R v_z R) + \cdots = 0$$
- Induction eq.– ϕ
  $$\partial_t B_\phi = \partial_z(\cdots) - \partial_R(v_R B_\phi - v_\phi B_R)$$

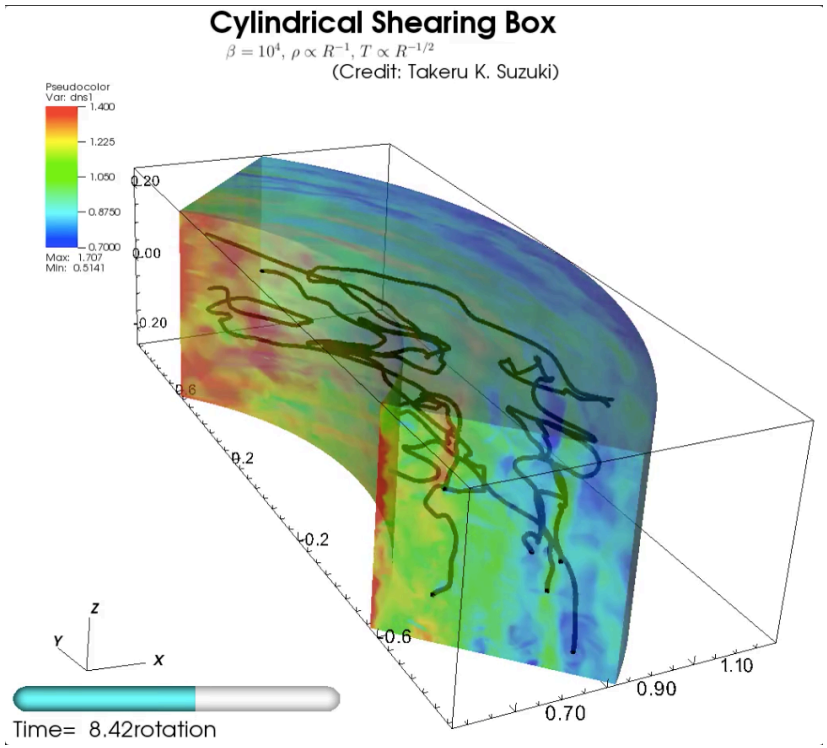
Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$
- Momentum– ϕ (Angular Momentum):
  $$\partial_t(\rho v_\phi R) + \partial_R[(\rho v_R v_\phi + B_R B_\phi/4\pi)R^2] + \cdots = 0$$
- Momentum– z:
  $$\partial_t(\rho v_z) + R^{-1}\partial_R(\rho v_R v_z R) + \cdots = 0$$
- Induction eq.– ϕ
  $$\partial_t B_\phi = \partial_z(\cdots) - \partial_R(v_R B_\phi - v_\phi B_R)$$
- Induction eq.– z
  $$\partial_t B_z = R^{-1}\partial_R[(v_z B_R - v_R B_z)R] - \partial_\phi(\cdots)$$

Equations

- Mass:
  $$\partial_t \rho + R^{-1}\partial_R(\rho v_R R) + \partial_\phi(\cdots) + \partial_z(\cdots) = 0$$
- Momentum– R:
  $$\partial_t(\rho v_R) + R^{-1}\partial_R(\rho v_R^2 R) + \cdots = 0$$
- Momentum– ϕ (Angular Momentum):
  $$\partial_t(\rho v_\phi R) + \partial_R[(\rho v_R v_\phi + B_R B_\phi/4\pi)R^2] + \cdots = 0$$
- Momentum– z:
  $$\partial_t(\rho v_z) + R^{-1}\partial_R(\rho v_R v_z R) + \cdots = 0$$
- Induction eq.– ϕ
  $$\partial_t B_\phi = \partial_z(\cdots) - \partial_R(v_R B_\phi - v_\phi B_R)$$
- Induction eq.– z
  $$\partial_t B_z = R^{-1}\partial_R[(v_z B_R - v_R B_z)R] - \partial_\phi(\cdots)$$
- $\nabla \cdot B = 0$
  $$R^{-1}\partial_R(B_R R) + R^{-1}\partial_\phi B_\phi + \partial_z B_z = 0$$

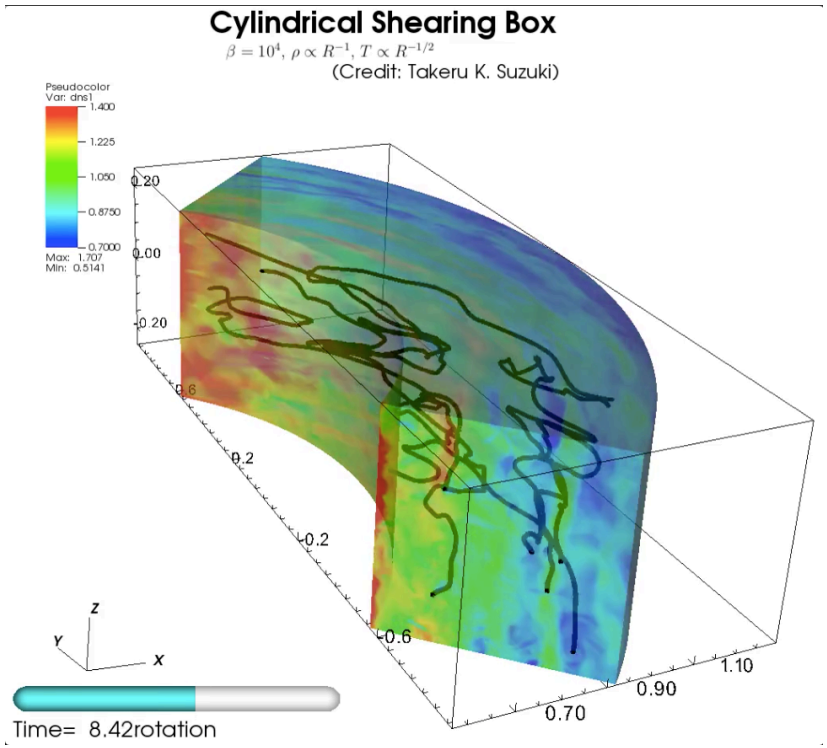
Cylindrical Shearing Condition



Cylindrical Shearing Condition

- Shear: $A(R_{\pm}, \phi, z) = A(R_{\mp}, \phi \pm \Delta\Omega_{\mathrm{eq}} t, z)$

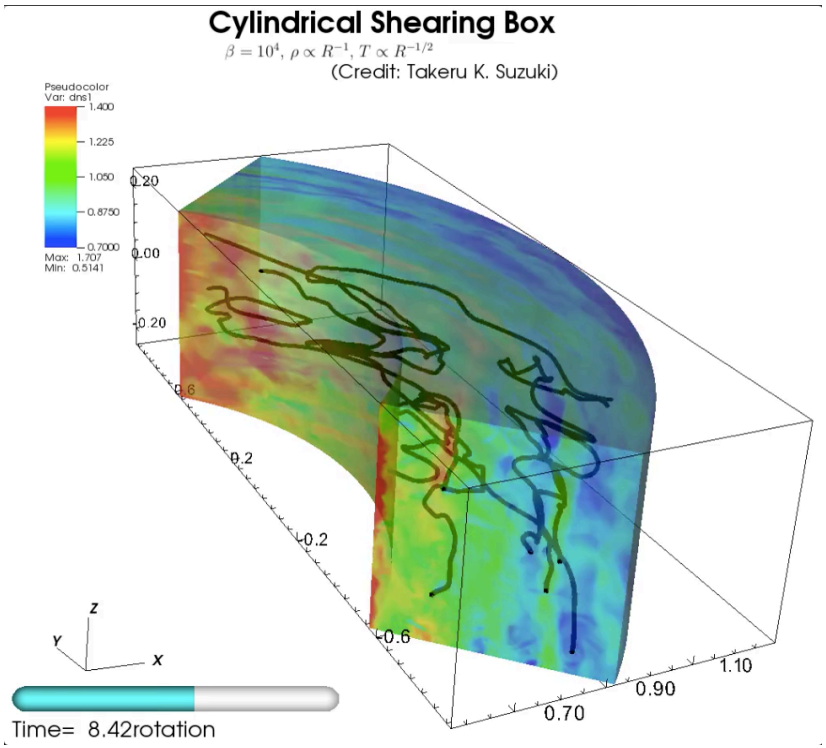
  where $\Delta\Omega_{\mathrm{eq}} = \Omega_{\mathrm{eq},-} - \Omega_{\mathrm{eq},+}$



Cylindrical Shearing Condition

- Shear: $A(R_\pm, \phi, z) = A(R_\mp, \phi \pm \Delta\Omega_{\rm eq} t, z)$

  where $\Delta\Omega_{\rm eq} = \Omega_{\rm eq,-} - \Omega_{\rm eq,+}$
- Conserved quantities, A, at R_- & R_+



$$A = \begin{cases} \rho v_R R \\ \rho v_R^2 R \\ (\rho v_R v_\phi + B_\phi B_R / 4\pi) R^2 \\ \rho v_R v_z R \\ v_R B_\phi - v_\phi B_R \\ (v_z B_R - v_R B_z) R \\ B_R R \end{cases}$$

Cylindrical Shearing Box (CySB)

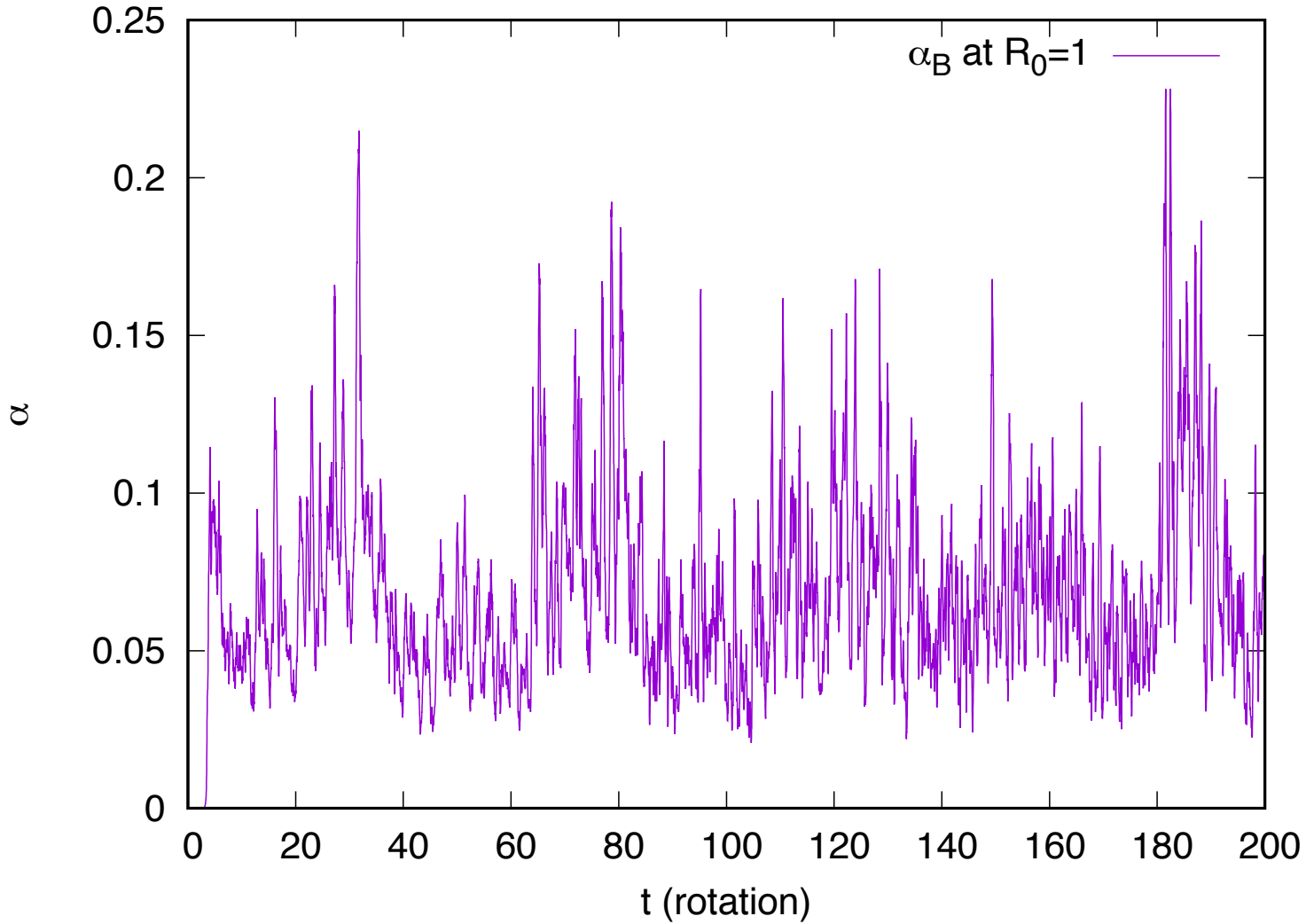
Cylindrical Shearing Box (CySB)

Time Evolution

$(\beta_{z,0} = 10^3,\ T \propto R^{-1/2}$, initial $\rho \propto R^{-1})$

$(L_R, L_\phi, L_z) = (0.4, \pi/6, 0.1)$ ≈(4H,4H,H) resolved by (256,256,64)

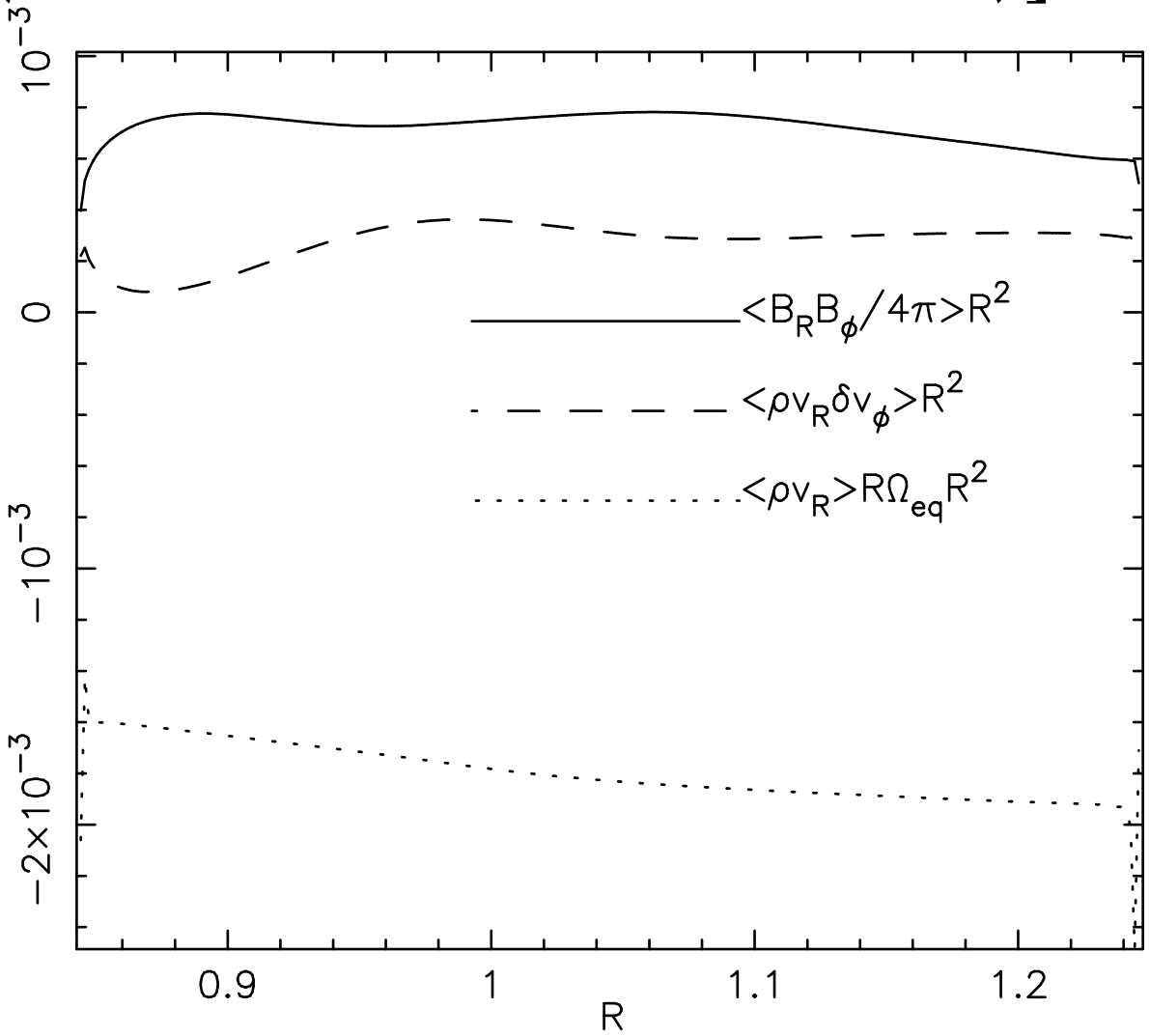
$$\alpha_{\mathrm{B}} = B_R B_\phi / 4\pi p$$



Angular Momentum Flows

Under the steady-state condition:

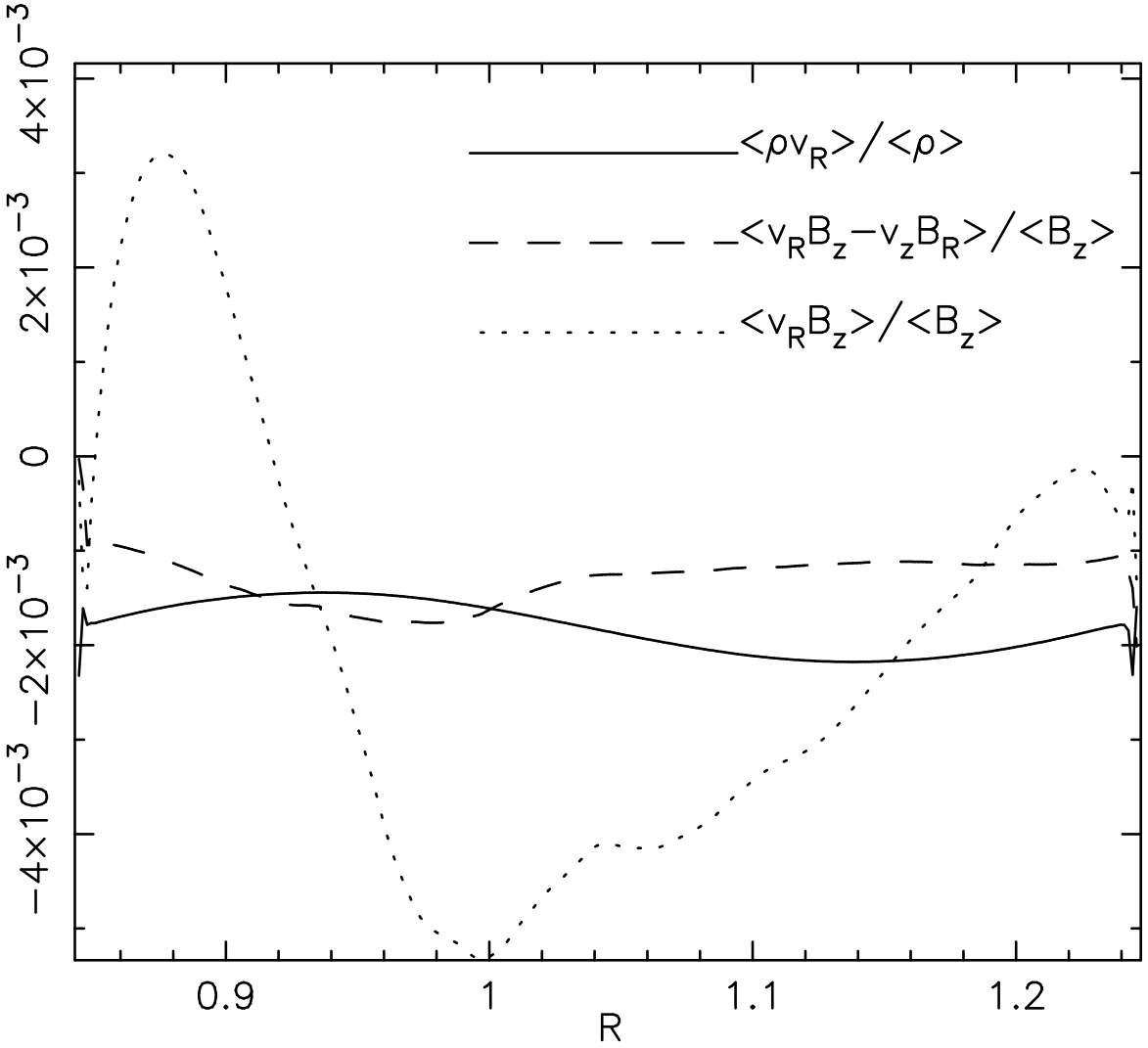
$$\frac{\partial}{\partial R}\left[R^2\left(\rho v_R R\Omega_{\mathrm{eq}} + \rho v_R \delta v_\phi - B_R B_\phi/4\pi\right)\right] = 0$$



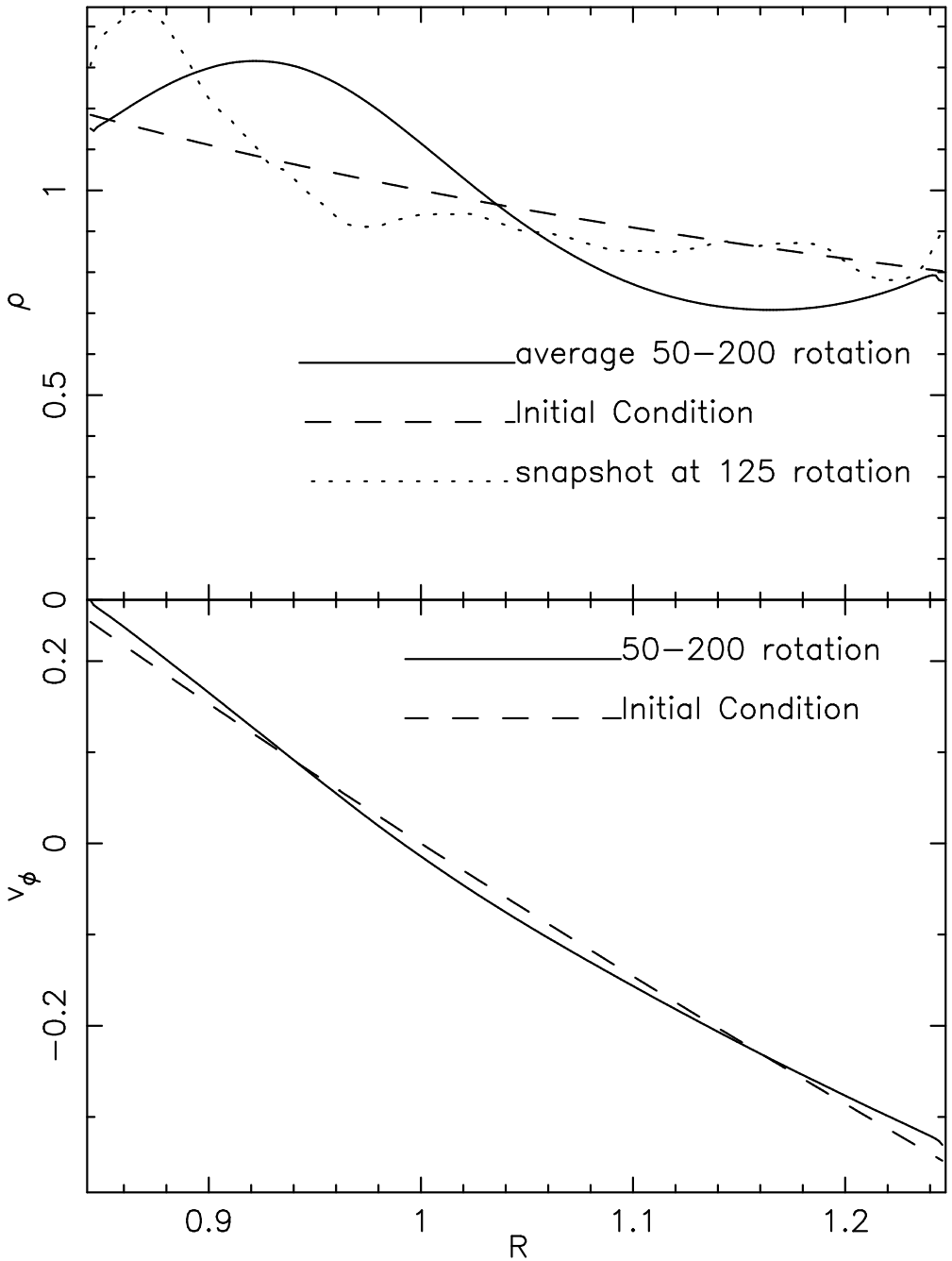
Accretion & B_z Advection

$$\frac{\partial}{\partial t}(RB_z) = \frac{\partial}{\partial R}\left[R\left(v_z B_R - v_R B_z\right)\right]$$

$$\Rightarrow \langle v_{R,B_z}\rangle = \langle R(v_R B_z - v_z B_R)\rangle / \langle RB_z\rangle$$



Zonal Flow



a long-lived ρ bump

Johansen+ 2009

- Physical ?
- Numerical Artifact ?

Summary

Cylindrical Shearing Box

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Various potential applications

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Various potential applications

- Accretion vs. $\boldsymbol{B}_z$ advection

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Various potential applications

- Accretion vs. $\boldsymbol{B}_z$ advection
- Particle Acceleration

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Various potential applications

- Accretion vs. $\boldsymbol{B}_z$ advection
- Particle Acceleration
- Dust grains in Prtoplanetary Disks

Summary

Cylindrical Shearing Box

Advantage to Cartesian SB

- can handle net mass accretion

Advantage to Global Simulations

- can resolve fine-scale turbulence
- can perform long-time simulations

Various potential applications

- Accretion vs. $\boldsymbol{B}_z$ advection
- Particle Acceleration
- Dust grains in Prtoplanetary Disks
- .....