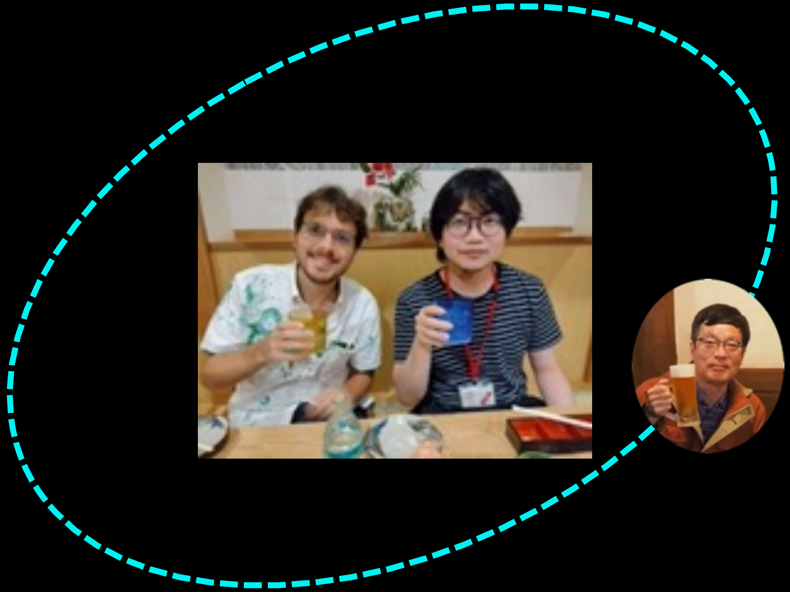
# Lagrange instability timescales of three-body systems



Hayashi, Trani & YS:
ApJ 939(2022)81
ApJ 943(2023)58

Yasushi Suto Department of Physics and Research Center for the Early Universe, The University of Tokyo

14:00-15:00 February 13, 2023 @Niels Bohr Institute

# 1 three-body dynamics: old and new problem

# Celestial mechanics and three-body problems = "superstring theory" before the 20th century

- Great mathematicians had seriously worked on the topic
  - Joseph-Louis Lagrange (1736-1813)
  - Pierre-Simon Laplace (1749-1827)
  - Johann Carl Friedrich Gauß (1777-1855)
  - Carl Gustav Jacob Jacobi (1804-1851)
  - William Rowan Hamilton (1805-1865)
  - Jules-Henri Poincaré (1854-1912)
- If quantum theory and relativity had not been discovered, three-body problem could have been the frontier in mathematical physics that the best scientists would choose
  - what can "we" do then? $\Rightarrow$ amazing three-body systems + accurate numerical simulations, which great people never expected in their epochs

# Ubiquity of hierarchical triples

- Stellar systems
  - more than 70% of OBA-type stars and 50% of FGK-type stars a belong to binary/multiple systems (e.g., Alpha Centauri)
- (Exo)Planetary systems
  - planets around binary stars, multi-planets, satellites,,,
- Compact objects
  - Possible pathway towards binary BHs detected by GW
  - Binaries (stars, BHs) around a supermassive BH in galaxies
  - Triples of compact objects, e.g., pulsar-WD binary + tertiary WD (Ransom et al. 2014)

# Diversities triggered by triple dynamics

***Alpha Centauri was a triple system, two suns tightly orbiting one another, and a third, more remote, circling them both. What would it be like to live on a world with three suns in the sky?***

***— Carl Sagan "Contact"***

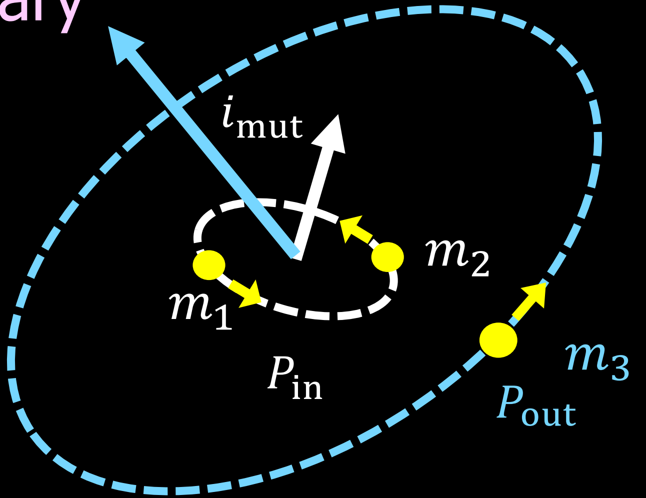




# 2 Hierarchical three-body systems

# Hierarchical three-body systems

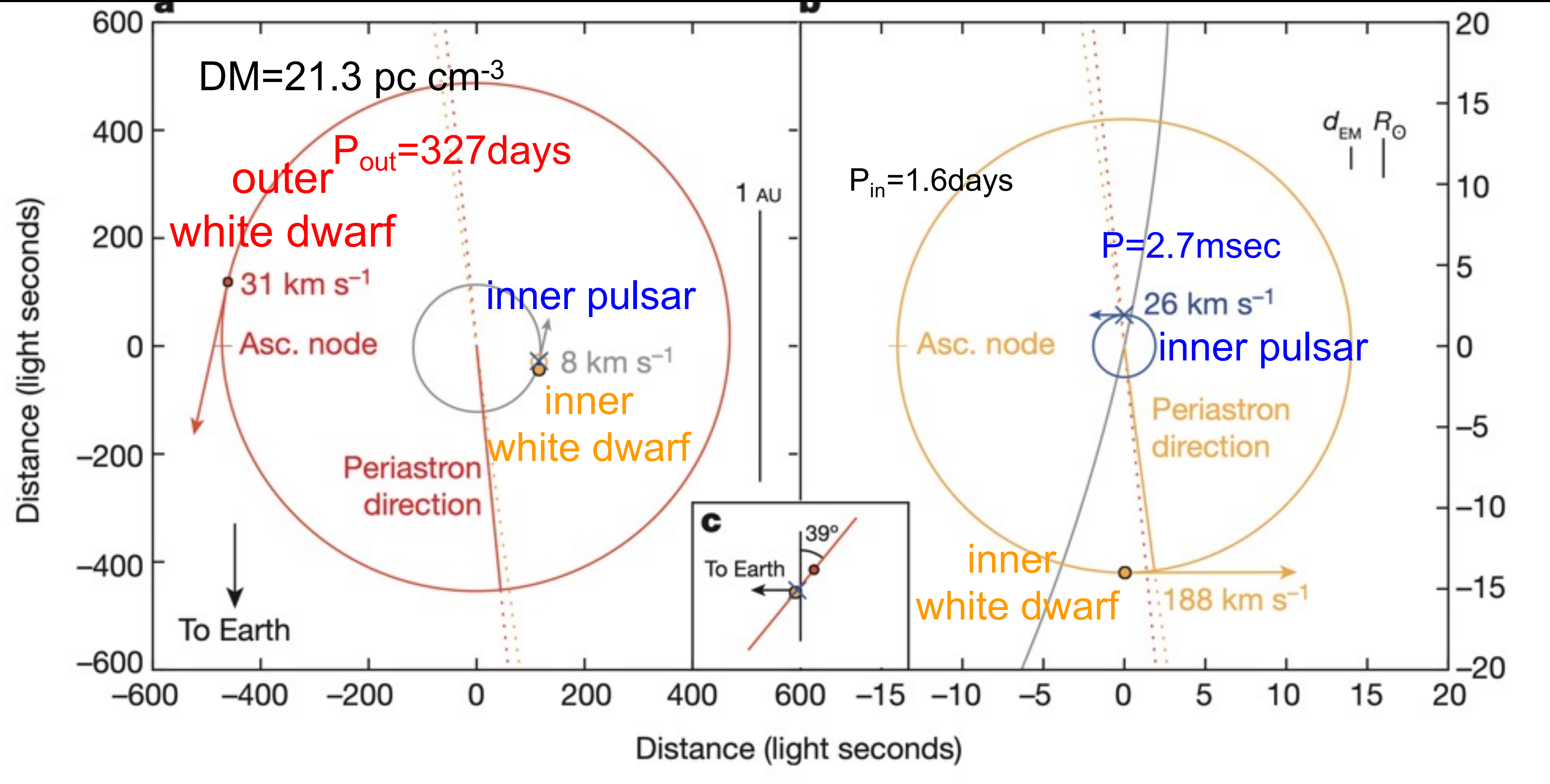
- Gravitational three-body systems are unstable in general
  - stable three-body systems are mostly hierarchical: tight binary + distant tertiary orbiting the center-of-mass of the inner binary
  - observed three-body systems are likely to be hierarchical
- Stable systems are inevitably associated with (undemocratic) hierarchies
  - quite universal in biological, astronomical and social systems
    - quarks and leptons – atoms – molecules – DNAs – cells – organs – animals – villages – cities – nations – planets – stars – star clusters – galaxies – galaxy clusters – universe(s) – multiverse(s)



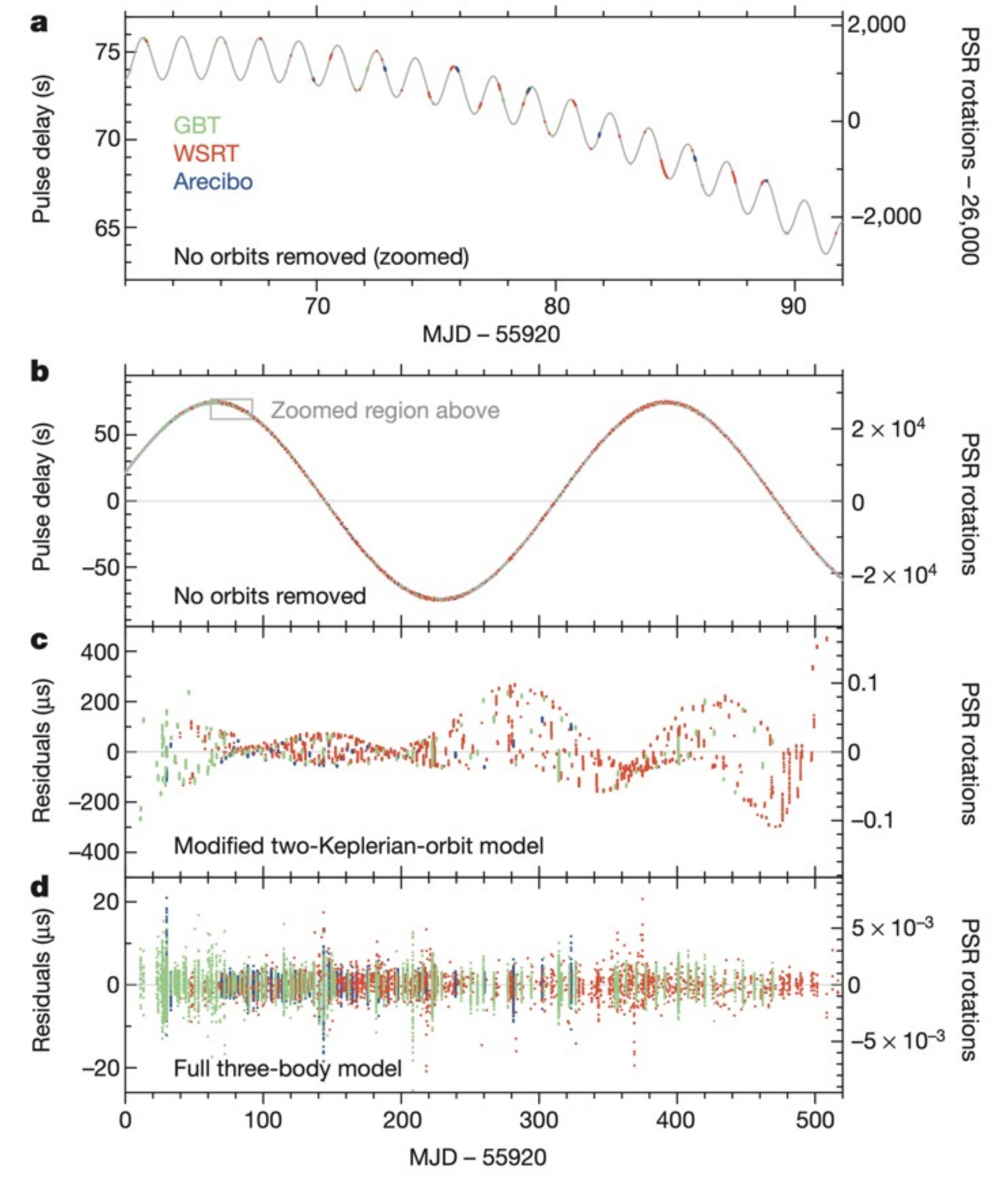

- non-intuitive (counter-intuitive) dynamical behavior of hierarchical triples triggers unexpectedly broad diversities in astronomical phenomena (e.g., ZKL effect)

# A millisecond pulsar in a stellar triple system

S. M. Ransom[1], I. H. Stairs[2], A. M. Archibald[3,4], J. W. T. Hessels[3,5], D. L. Kaplan[6,7], M. H. van Kerkwijk[8], J. Boyles[9,10], A. T. Deller[3], S. Chatterjee[11], A. Schechtman-Rook[7], A. Berndsen[2], R. S. Lynch[4], D. R. Lorimer[9], C. Karako-Argaman[4], V. M. Kaspi[4], V. I. Kondratiev[3,12], M. A. McLaughlin[9], J. van Leeuwen[3,5], R. Rosen[1,9], M. S. E. Roberts[13,14] & K. Stovall[15,16]




**Ransom et al. Nature 505 (2014) 520**

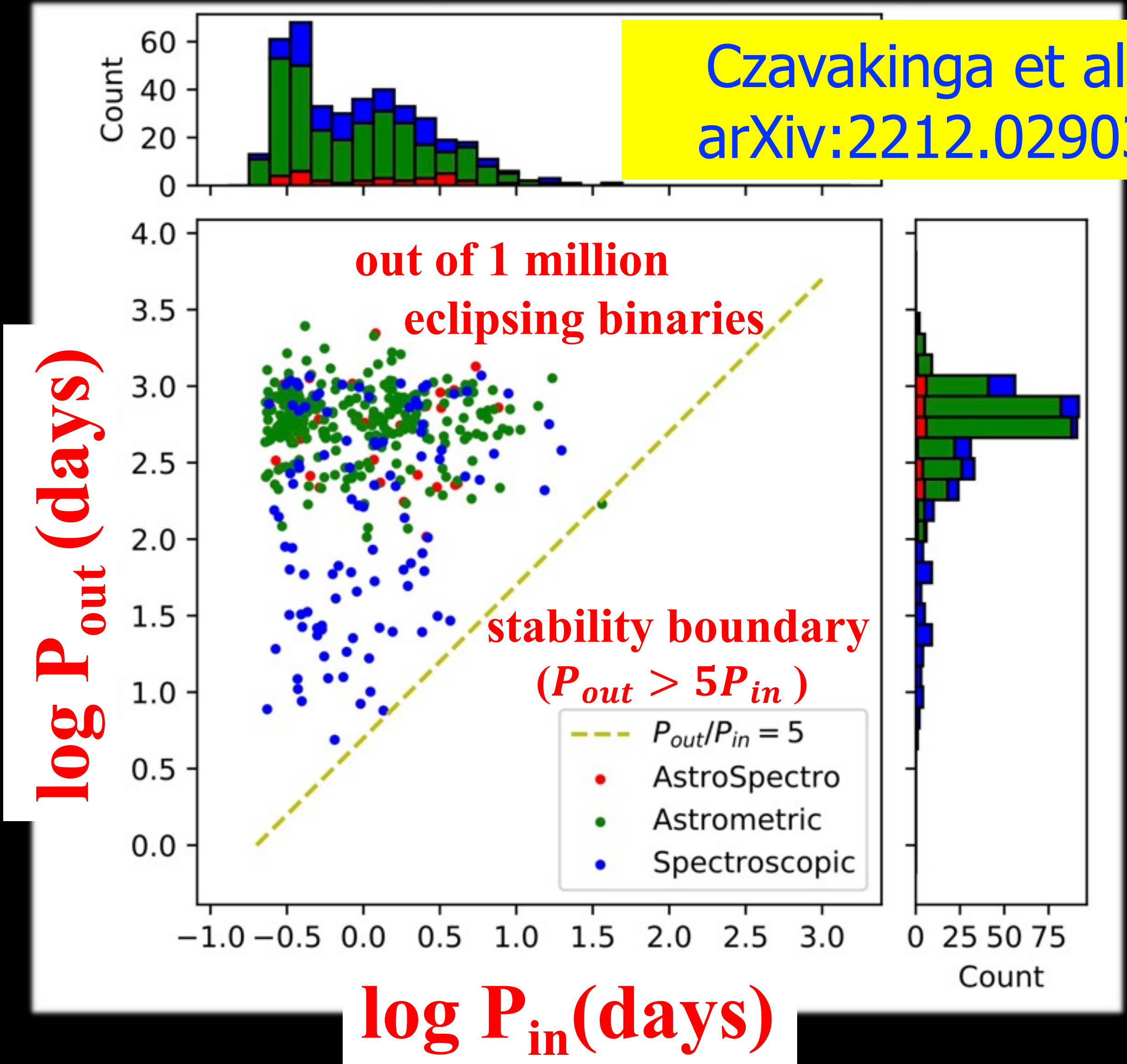


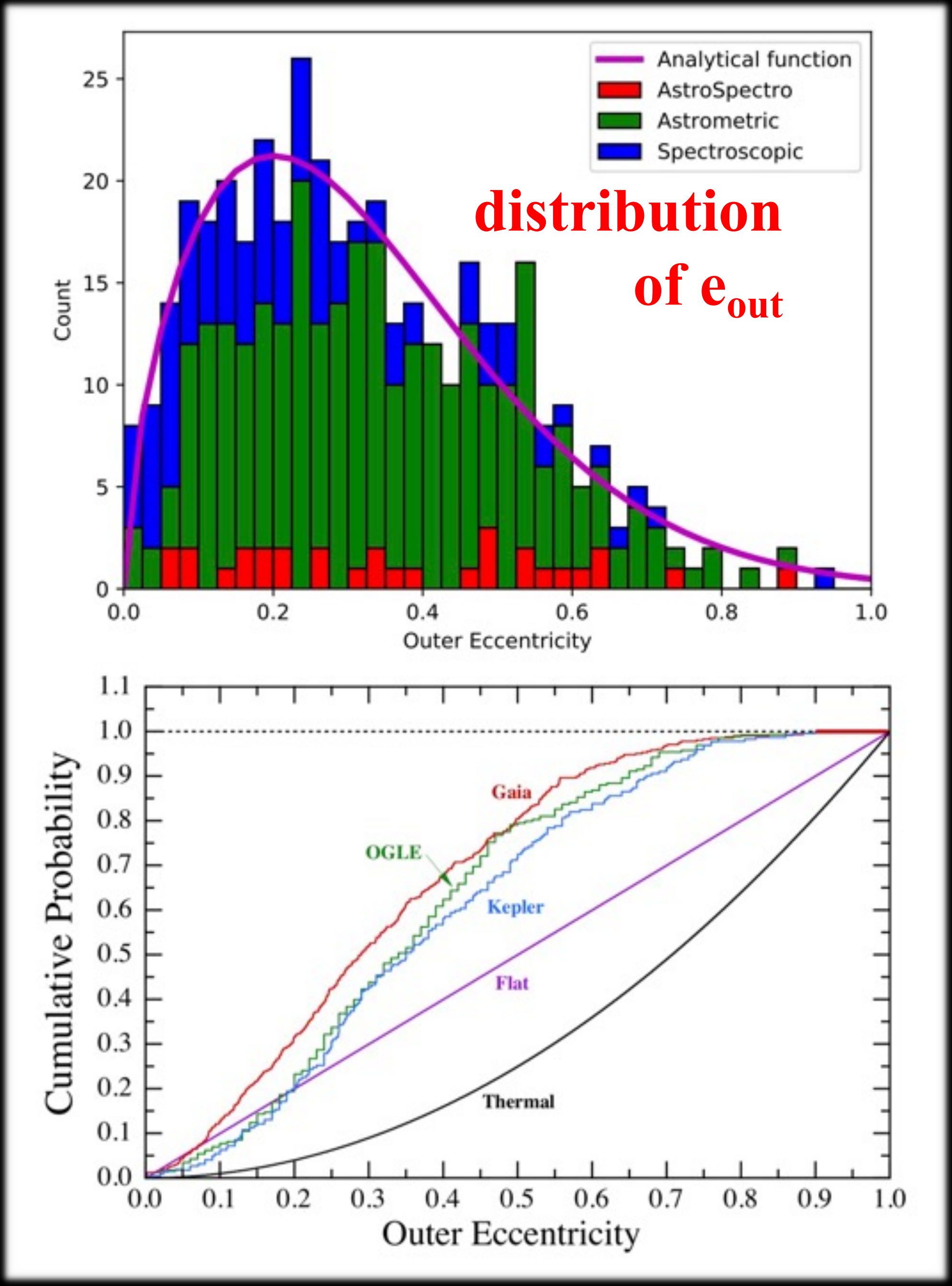

# Ransom et al.
# Nature 505(2014)520
# NS-WD binary + WD

| PSR J0337+1715 parameters | |
|---|---|
| inner orbital period (pulsar+WD) | 1.629401788(5) day |
| outer orbital period (WD) | 327.257541(7) day |
| pulsar spin period | 2.73258863244(9) msec |
| mutual orbital inclination | 0.0120(17) deg. |
| Pulsar mass | 1.4378(13) $M_\odot$ |
| Inner WD mass | 0.19751(15) $M_\odot$ |
| Outer WD mass | 0.4101(3) $M_\odot$ |

# Hierarchical triple-star candidates in Gaia DR3 (403 in total)

Czavakinga et al.
arXiv:2212.02903

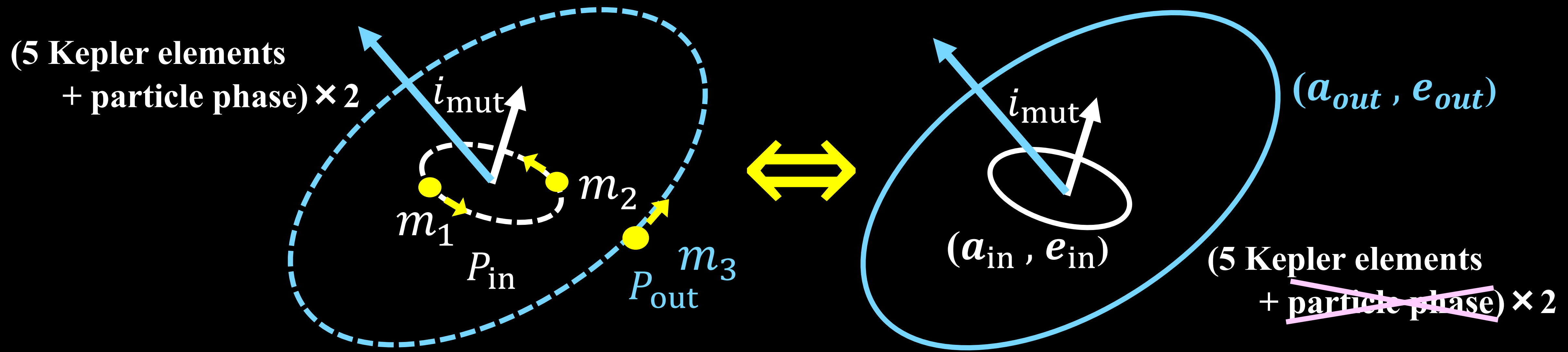


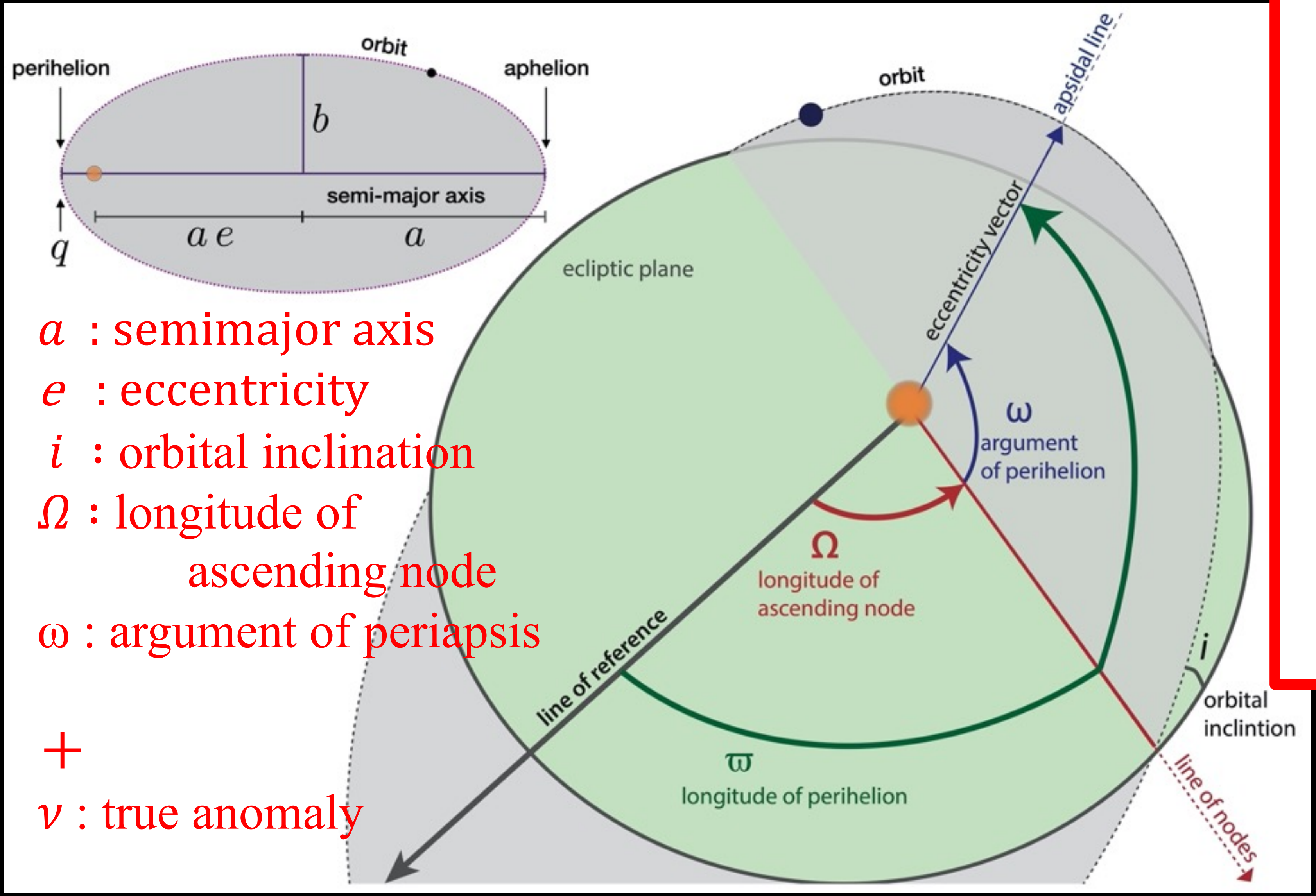
# 3 Three-body problem and von Zeipel-Lidov-Kozai effect

# Secular approximation to triple dynamics

- Very different timescales involved: $P_{in} \ll P_{out}$
  - time-consuming numerical integration
- perturbative expansion in terms of $a_{in}/\, a_{out} \ll 1$
  - long-time numerical integration by approximating the particle-particle interaction with the ring-ring interaction over appropriate time-averaging of particles on their orbits



# Kepler orbital elements




$a$ : semimajor axis

$e$ : eccentricity

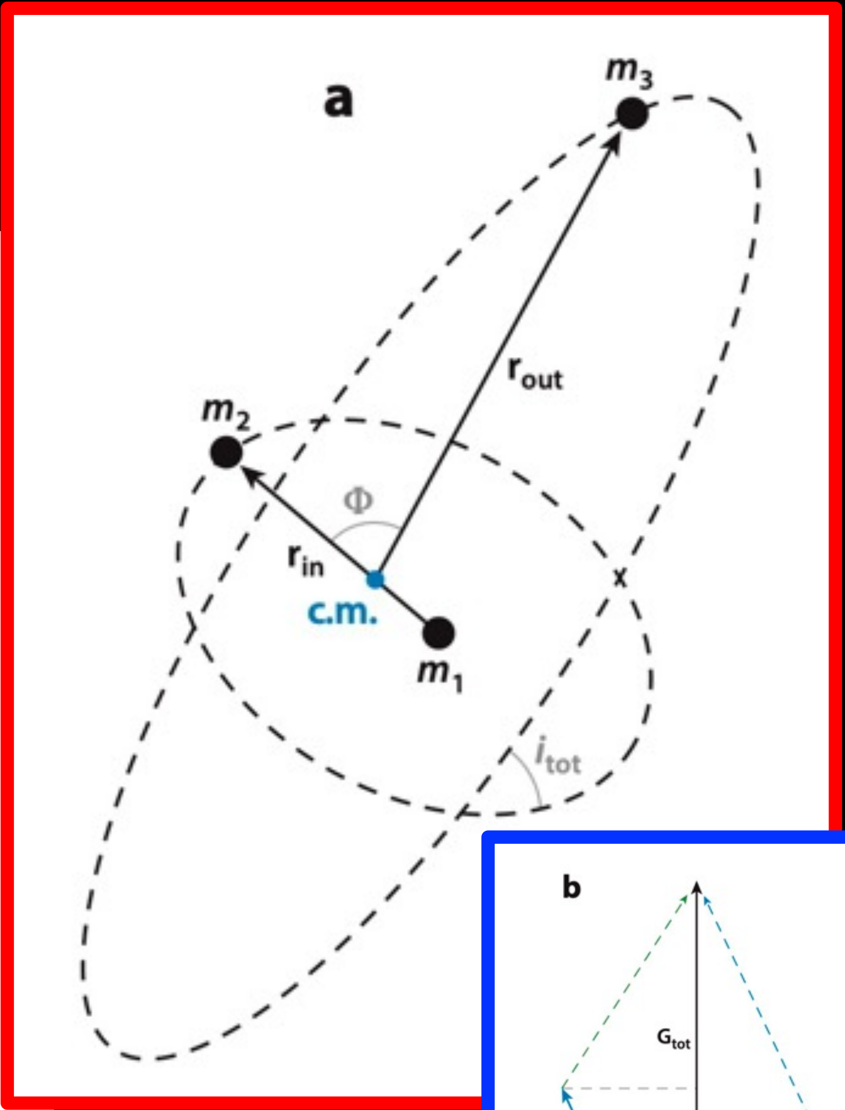
$i$ : orbital inclination

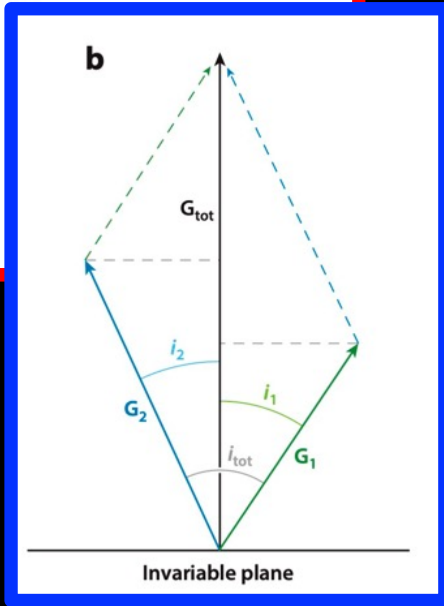
$\Omega$ : longitude of ascending node

ω : argument of periapsis

+

$\nu$ : true anomaly



# Legendre expansion of Hamiltonian

Kepler motion for inner orbit

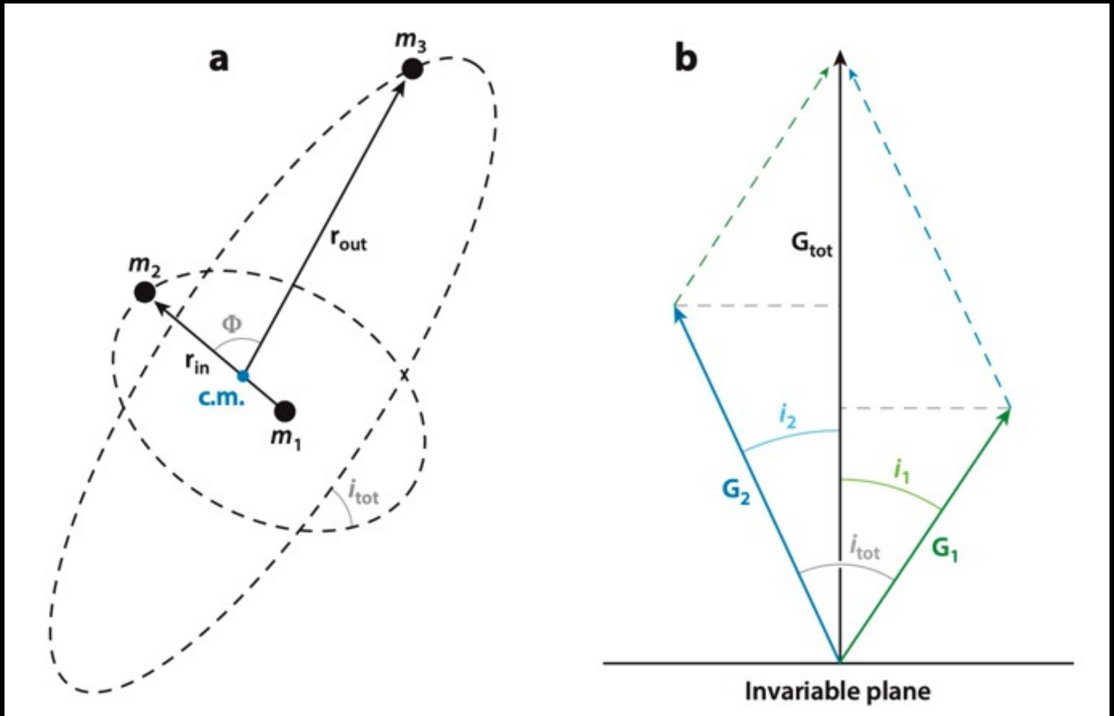
interaction between inner and outer orbits

$$\mathcal{H} = \boxed{\frac{k^2 m_1 m_2}{2a_1}} + \boxed{\frac{k^2 m_3 (m_1 + m_2)}{2a_2}} + \boxed{\frac{k^2}{a_2} \sum_{n=2}^{\infty} \left(\frac{a_1}{a_2}\right)^n M_n \left(\frac{\mathbf{r}_{\rm in}}{a_1}\right)^n \left(\frac{a_2}{\mathbf{r}_{\rm out}}\right)^{n+1} P_n(\cos\Phi)}$$

Kepler motion for outer orbit

coupling constant

$$\boxed{M_n = m_1 m_2 m_3 \frac{m_1^{n-1} - (-m_2)^{n-1}}{(m_1 + m_2)^n}}$$




$$\mathcal{H} = \mathcal{H}_{\rm Kep,in} + \mathcal{H}_{\rm Kep,out} + \mathcal{H}_{\rm int}$$

$$\mathcal{H}_{\rm int} = \mathcal{H}_{\rm quad} + \mathcal{H}_{\rm oct} + \cdots$$

⇒ approximation by double-averaging of the Hamiltonian over the inner and outer orbits

# double-averaged quadrupole and octupole Hamiltonians

$$\langle \mathcal{H}_{\rm quad} \rangle = \frac{\boxed{\mu_{12}\Phi_0}}{16}[(2+3e_1^2)(3\cos^2 i_{\rm tot} - 1) + 15e_1^2 \sin^2 i_{\rm tot} \cos 2\omega_1]$$

$$\langle \mathcal{H}_{\rm oct} \rangle = -\frac{15}{64}\boxed{\mu_{12}\Phi_0} e_1 \boxed{\varepsilon_{\rm oct}} [\boxed{A\cos\phi} + 10\cos i_{\rm tot} \sin^2 i_{\rm oct} (1-e_1^2) \sin\omega_1 \sin\omega_2]$$

$$\boxed{\varepsilon_{\rm oct} = \frac{m_1 - m_2}{m_1 + m_2}\frac{a_1}{a_2}\frac{e_2}{1-e_2^2}}$$

$$\boxed{\mu_{12}\Phi_0 = \frac{m_1 m_2}{m_1+m_2}\frac{G m_3 a_1^2}{a_2^3(1-e_2^2)^{3/2}}}$$

- octupole term vanishes if $m_1=m_2$ (equal-mass inner binary) or $e_2=0$ (circular outer orbit)
- both $a_{in}$ and $a_{out}$ are conserved in the secular approximation ⇒ no energy exchange between inner and outer orbits (angular momentum exchange alone)

$$\boxed{A = 4 + 3e_1^2 - \frac{5}{2}\sin^2 i_{\rm tot}(2 + 5e_1^2 - 7e_1^2 \cos 2\omega_1)}$$

$$\boxed{\cos\phi = -\cos\omega_1 \cos\omega_2 - \cos i_{\rm tot} \sin\omega_1 \sin\omega_2}$$

# von Zeipel-Lidov-Kozai effect

- Takashi Ito and Katsuhito Ohtsuka (2019) "The Lidov–Kozai Oscillation and Hugo von Zeipel" Monogr. Environ. Earth Planets, 7, 1–113
- von Zeipel, H. (1910) "Sur l'application des séries de M. Lindstedt à l'étude du mouvement des comètes périodiques", Astronomische Nachrichten, 183, 345–418
- M.L. Lidov (1961) "Evolution of the orbits of artificial satellites of planets as affected by gravitational perturbation from external bodies" Artificial Earth Satellite, 8, 5–45
- Kozai, Yoshihide (1962) "Secular perturbations of asteroids with high inclination and eccentricity" The Astronomical Journal, 67, 591–598
- Naoz, S. (2016) "The eccentric Kozai–Lidov effect and its applications", Annual Reviews of Astronomy and Astrophysics, 54, 441–489

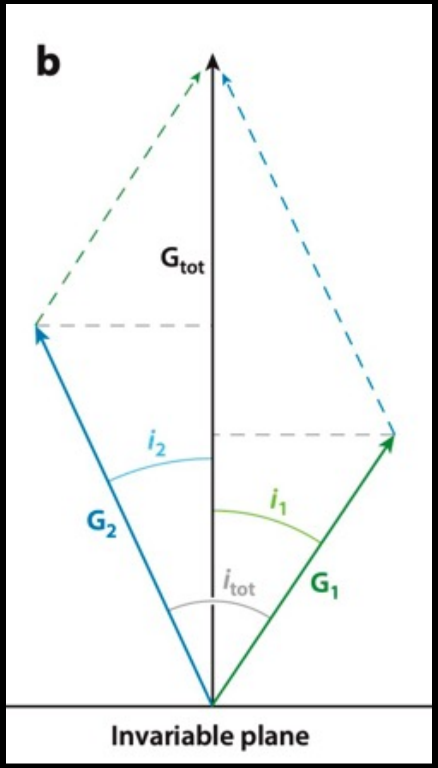
# standard and eccentric ZKL effects

- standard ZKL
  - test particle limit ($m_2$=0) and circular outer orbit ($e_{out}$=0)
  - angular momentum of the inner orbit along the total (outer) orbital axis is conserved

$$\boxed{j_{1,z} = \sqrt{1-e_1^2}\cos i_1 = \sqrt{1-e_{1,\mathrm{init}}^2}\cos i_{1,\mathrm{init}}}$$

  - inner eccentricity and inclination periodically change with

$$e_{1,\max} = \sqrt{1-\frac{5}{3}\cos^2 i_{1,\mathrm{init}}} \quad \text{if} \quad 39.2^\circ < i_{1,\mathrm{init}} < 140.8^\circ$$




- eccentric ZKL
  - $e_{out} \sim 1 \Rightarrow$ more drastic effect due to the octupole term

# examples of ZKL effects in secular approximation

Naoz, S: ARAA 54(2016)441

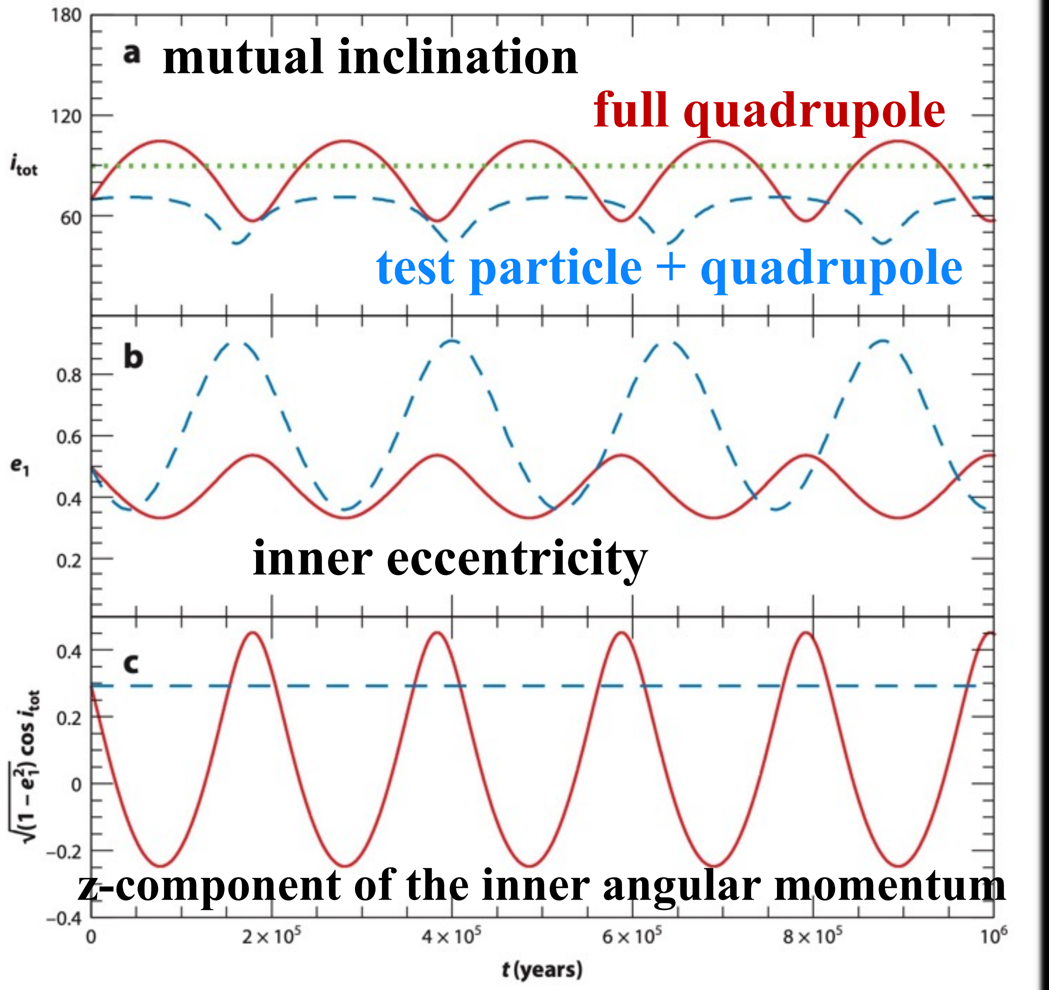



**Figure 3**

Comparison between the test particle quadrupole (TPQ) formalism (*dashed blue lines*) and the full quadrupole calculation (*solid red lines*). The system has an inner binary with $m_1 = 1.4 M_\odot$ and $m_2 = 0.3\ M_\odot$, and the outer body has mass $m_3 = 0.01\ M_\odot$. The orbit separations are $a_1 = 5$ AU and $a_2 = 50$ AU. The system was set initially with $e_1 = 0.5$ and $e_2 = 0$, $\omega_1 = 120°$ and $\omega_2 = 0$, and relative inclination $i_{tot} = 70°$. The panels

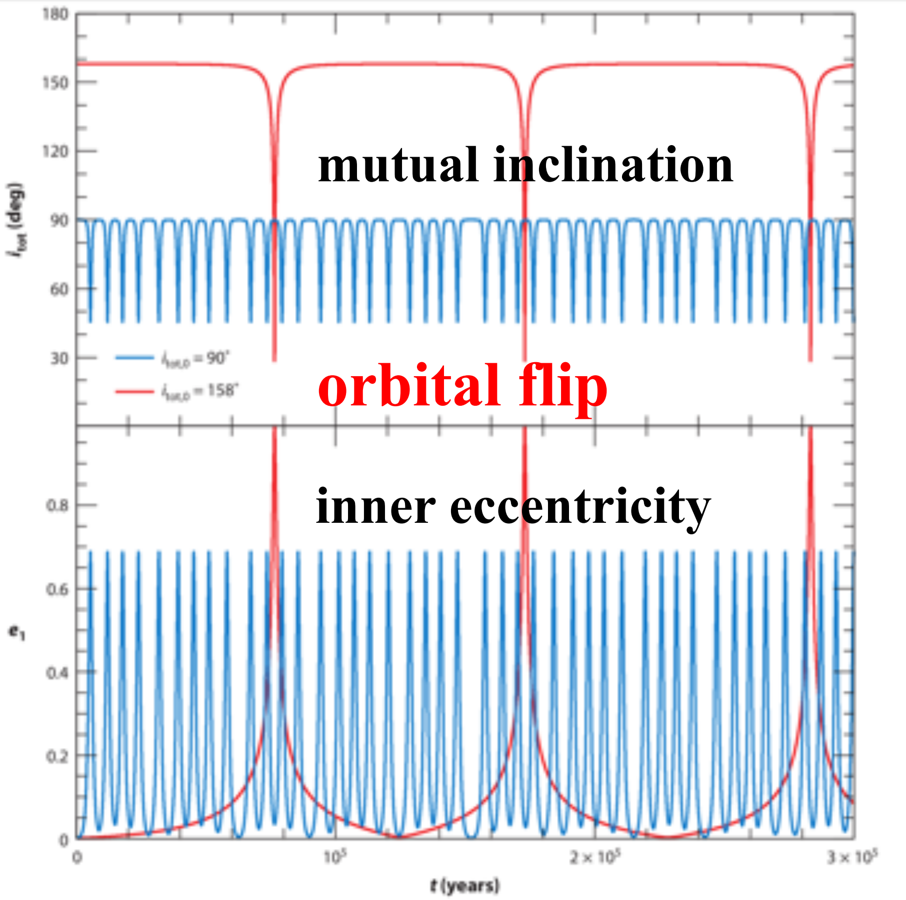
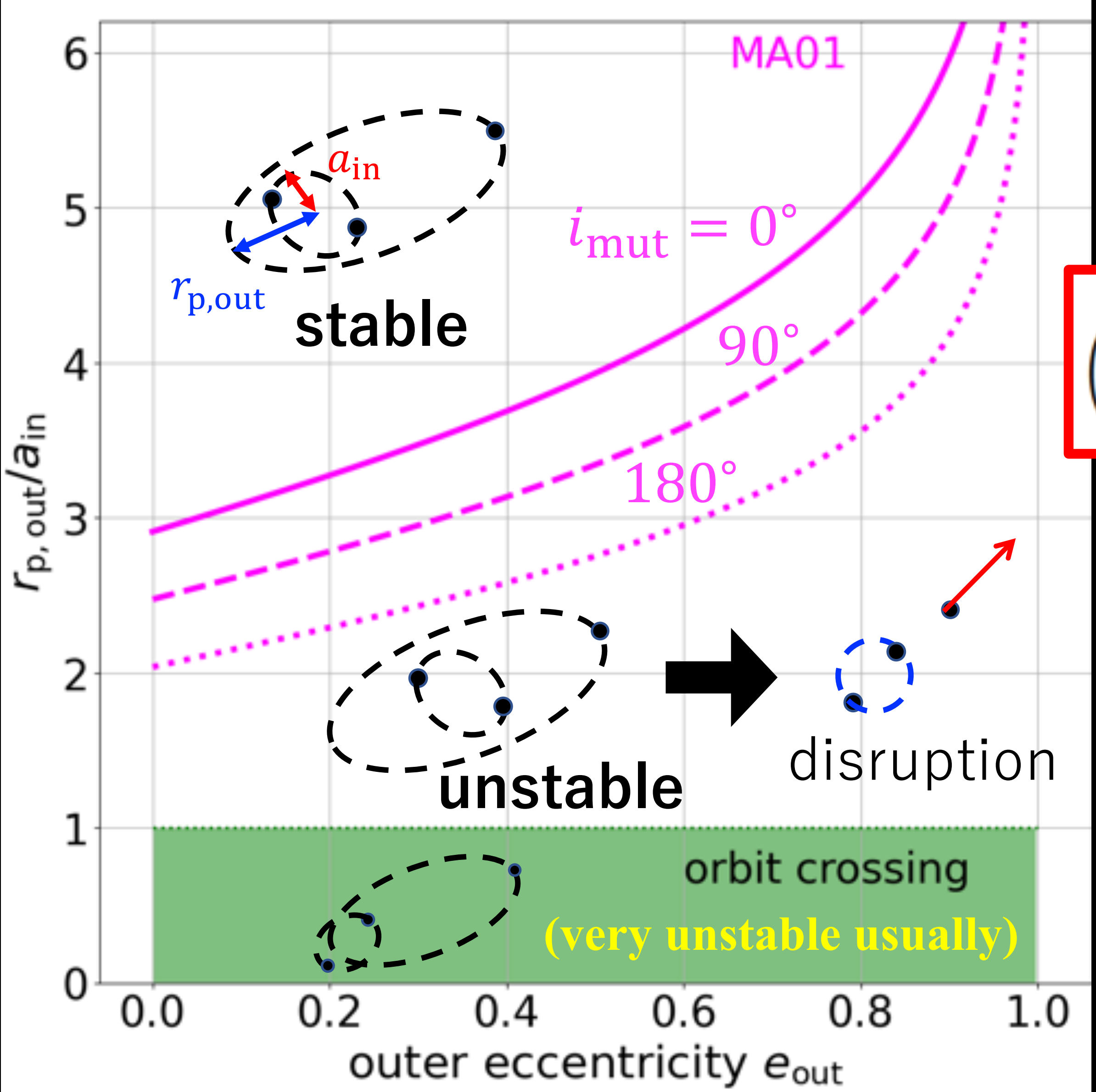



**Figure 4**

Small-mass outer perturber that induces large eccentricity excitation away from the nominal range of the Kozai angles of 39.2°–140.77°. We consider $m_1 = 1\ M_\odot$, $m_2 = 0.5\ M_\odot$, $m_3 = 0.05\ M_\odot$, $a_1 = 0.5$ AU, and $a_2 = 5$ AU. Both outer and inner eccentricities are set initially to zero, and also set initially are $\omega_1 = 90°$ and $\omega_2 = 0°$. We show two examples: The first shows the eccentricity excitations for as expected initial mutual inclination of $i_{tot} = 90°$, where in this case $i_1 = 25.01°$ and $i_2 = 64.99°$. This produces eccentricity excitation with $e_{1,max} = 0.689$. We also consider an example for which the mutual inclination is set initially to be $i_{tot} = 158°$. In this case $i_1 = 17.12°$ and $i_2 = 140.88°$. The latter parameters are adapted from Martin & Triaud (2015b), which leads to maximum inner eccentricity of $e_{1,max} = 0.99$. Note that in both examples $i_2$ is close to the nominal Kozai angles range.

# 4 Lyapunov vs. Lagrange instabilities of triple systems

# Triples are unstable in general ⇔ diversity




- Stability criterion
  (Mardling & Aarseth 2001)

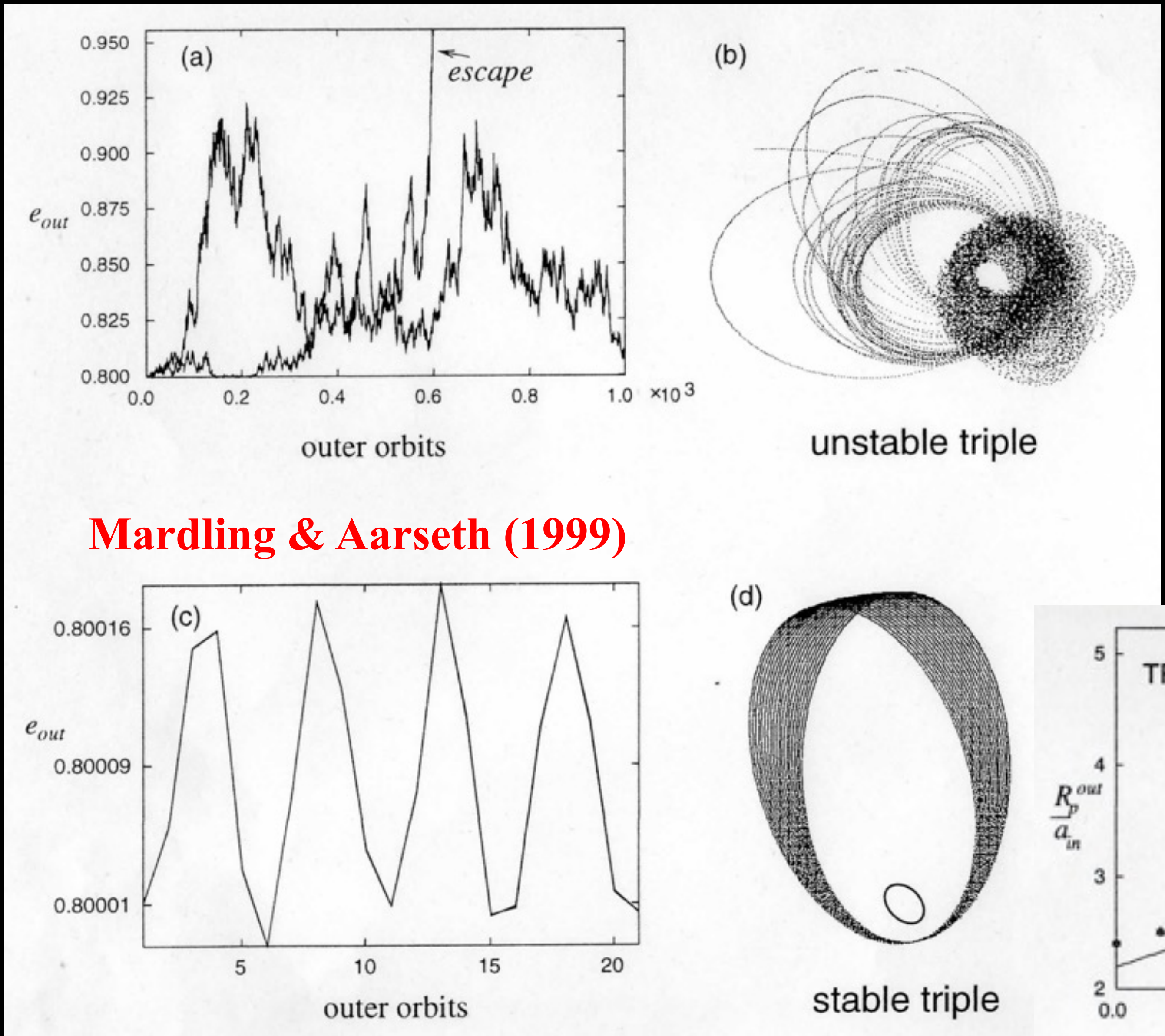
$$\boxed{\left(\frac{r_{\rm p,out}}{a_{\rm in}}\right)_{\rm MA} \equiv 2.8\left(1-0.3\frac{i_{\rm mut}}{\pi}\right)\left[\left(1+\frac{m_3}{m_{12}}\right)\frac{(1+e_{\rm out})}{\sqrt{1-e_{\rm out}}}\right]^{2/5}}$$

  - Well-known and widely used, but its implication is often misinterpreted…
  - What does it mean?
  - Lyapunov (chaoticity of local trajectory) vs. Lagrange (escape of a body from the system) stability
  - What is the disruption timescale?

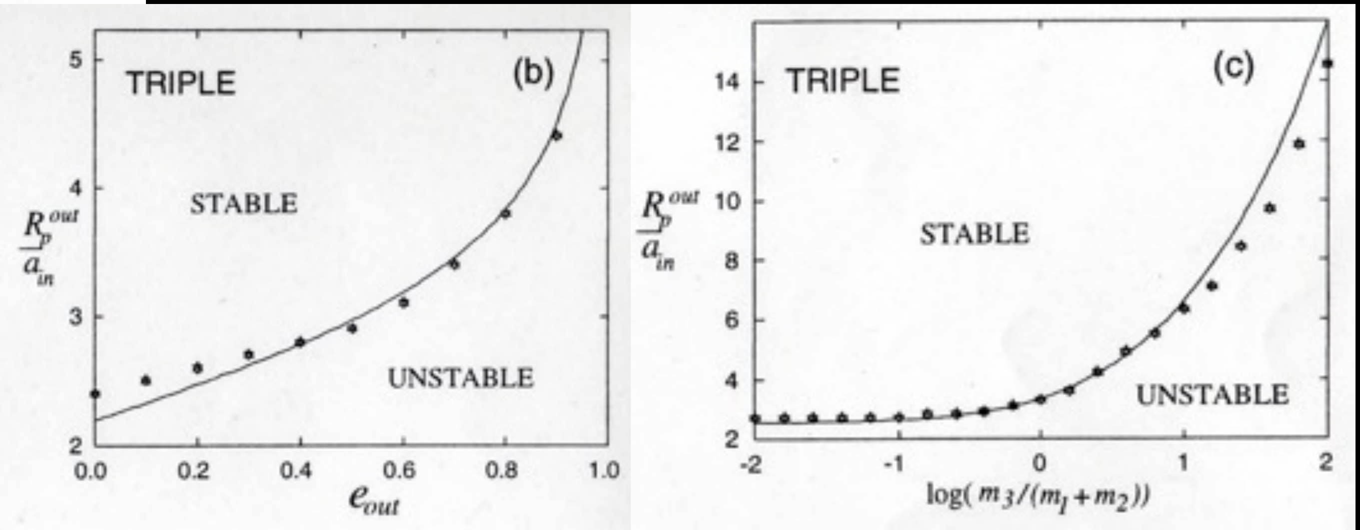
# Normal and chaotic evolution of triple systems

"A system was deemed stable if two orbits, initially differing by 1 part in $10^5$ in the eccentricity, remained close after 100 orbits." (Mardling & Aarseth 1999)

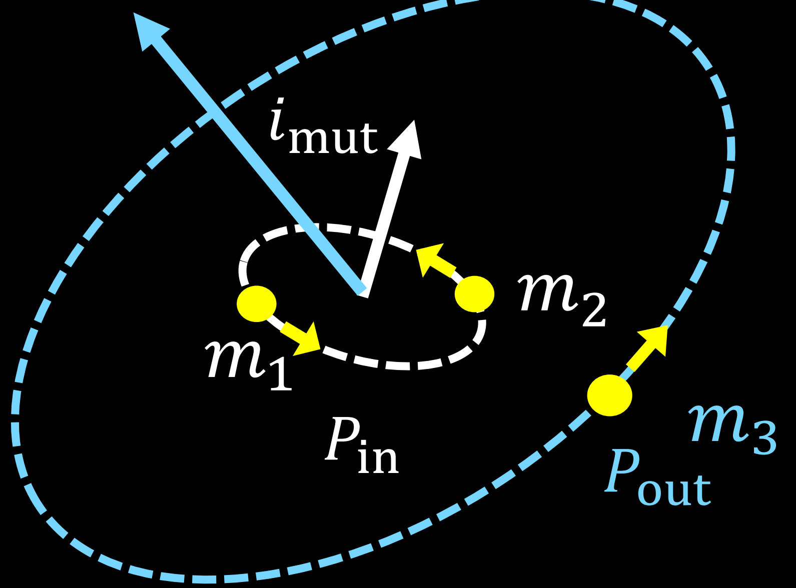
"We deem a triple system stable if it remains bound for 100 outer orbits and if the semimajor axes of both inner and outer orbits do not change by more than 10 percent of the initial value." (Vynatheya et al. 2022 MNRAS 516, 4146 )




Mardling & Aarseth (1999)

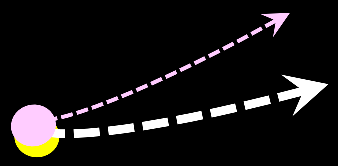


# Lyapunov vs. Lagrange instability

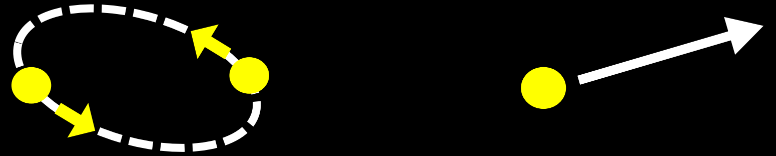



- cannot be studied under secular approximation (energies of inner and outer orbits are conserved separately).
- Relation between Lyapunov and Lagrange instabilities is not clear

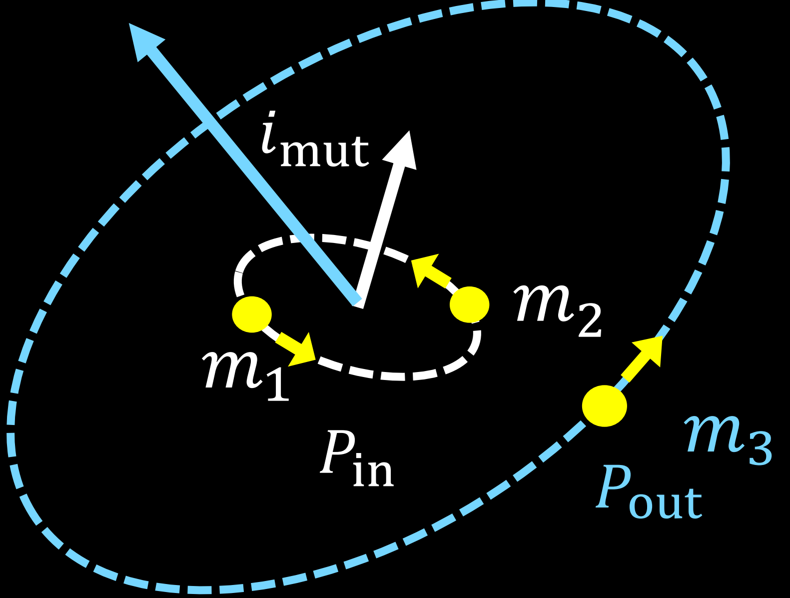
- Lyapunov instability
  - Local divergence of trajectories of bodies (~chaoticity)



- Lagrange instability
  - global escape of a body from the triple system (boundedness of an orbit)



# Longer-term N-body simulations in Newtonian dynamics neglecting GR effects




$e_{\rm in} = 10^{-5}$(circular)

Hayashi, Trani & YS,
arXiv:2207.12672 ApJ 939(2022)81
arXiv:2209.08487 ApJ in press

**N-body code TSUNAMI (Trani and Spera)**

$\frac{r_{\rm p,out}}{a_{\rm in}}$ $e_{\rm out}$

Integration time up to $10^9 P_{\rm in}$ (roughly $10^{6\text{-}7}$ $P_{\rm out}$)

Disruption time

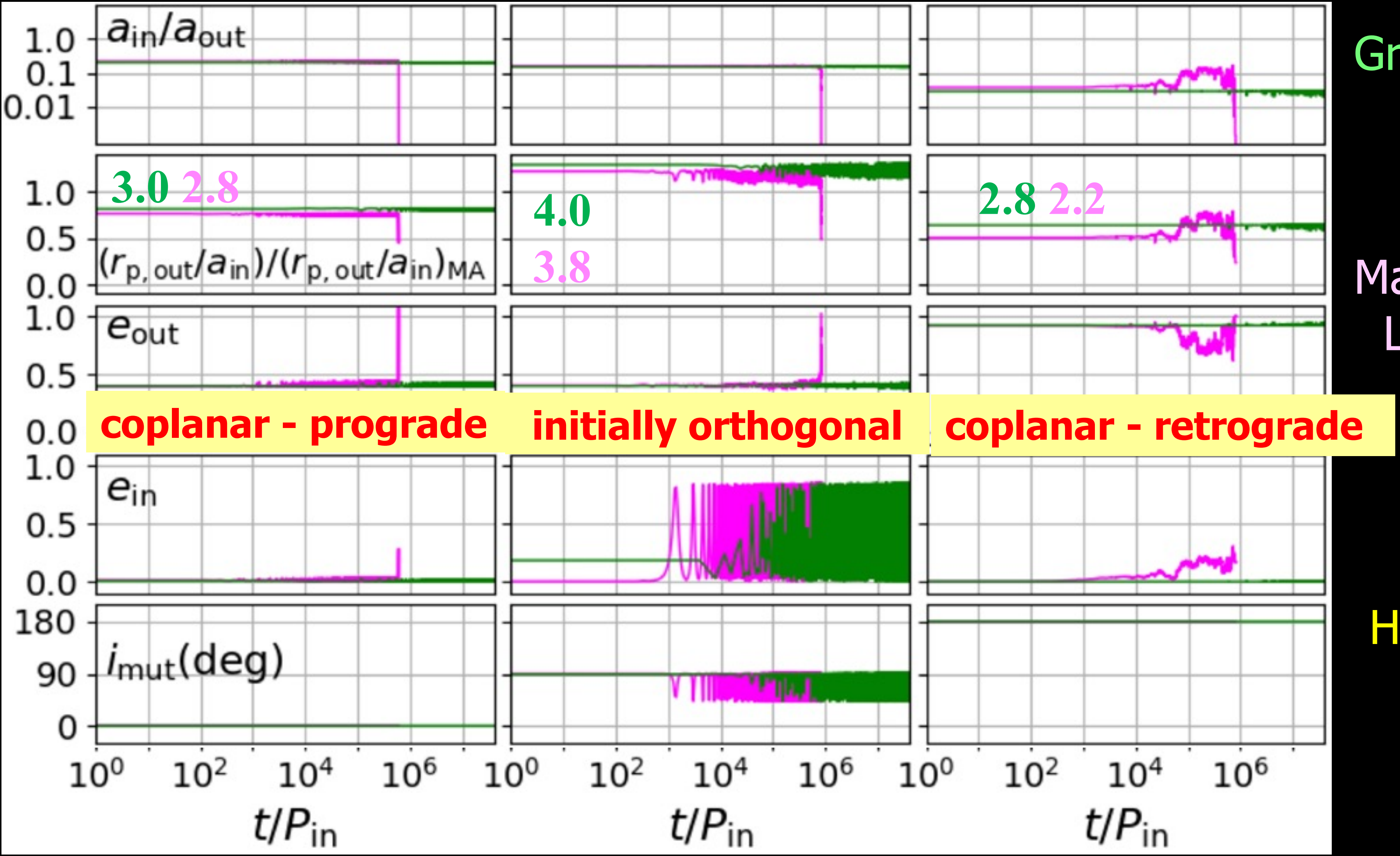
$$\frac{T_{\rm d}}{P_{\rm in}}(q_{21}, q_{23}, i_{\rm mut}, \cdots)$$

$i_{\rm mut}$: $0^\circ, 90^\circ, 180^\circ$ Coplanar: prograde, retrograde; Orthogonal

$q_{21} \equiv m_2/m_1 (\le 1)$ Mass ratio (inner binary)

$q_{23} \equiv m_2/m_3$ Mass ratio (tertiary)

# Examples of orbital evolution of triples



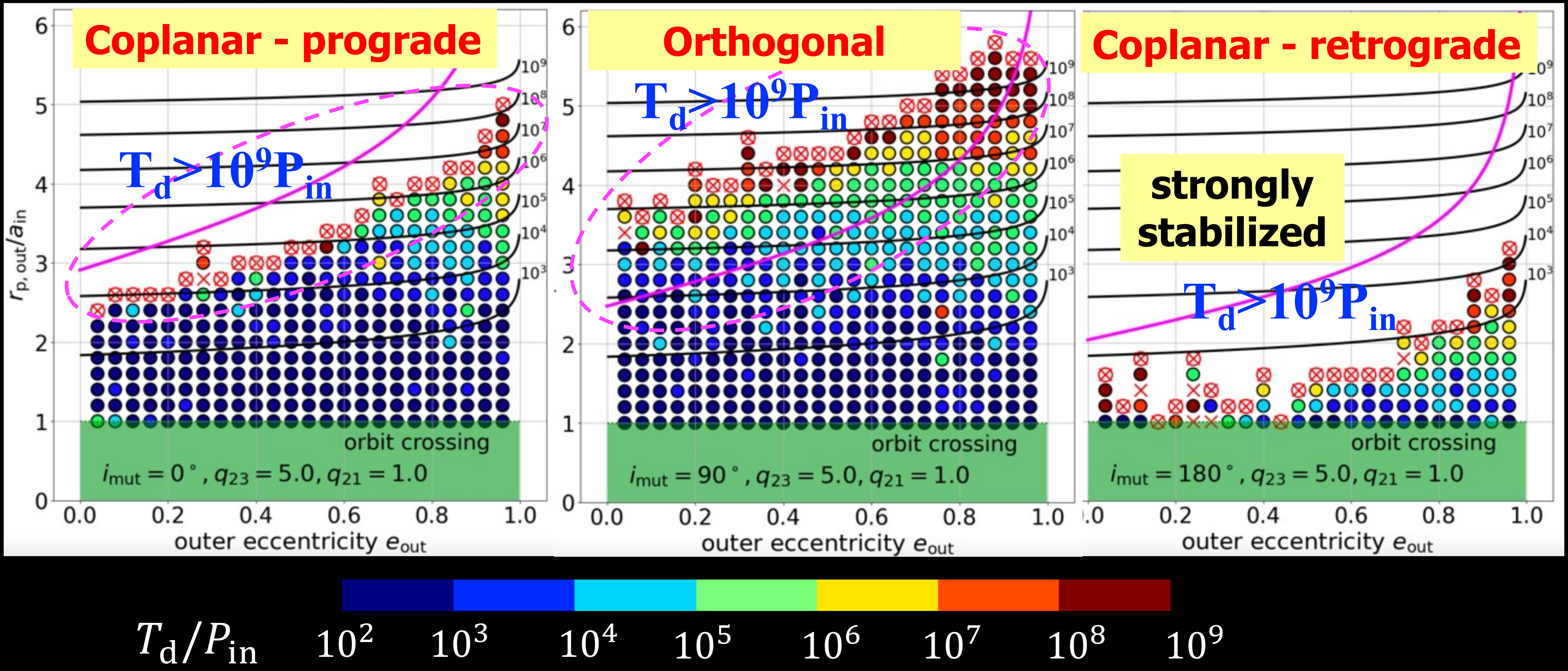

Green:
Lagrange stable up to $t=10^8$ $P_{in}$

Magenta:
Lagrange unstable around $t=10^6$ $P_{in}$

Hayashi, Trani & YS
arXiv:2207.12672
ApJ 939 (2022)81

# Disruption timescales on $a_{out}(1-e_{out})/a_{in}$ – $e_{out}$ plane

**$m_1=m_2=5m_3$** —— Dynamical stability boundary by Mardling & Aarseth (2001)



# Inclination dependence

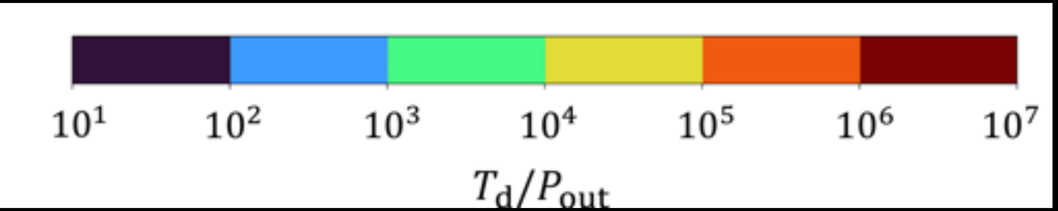
## $T_d/P_{out}$ on $a_{out}(1-e_{out})/a_{in}$ – $e_{out}$ plane

$60<i_{mut}$ (deg)$<150$

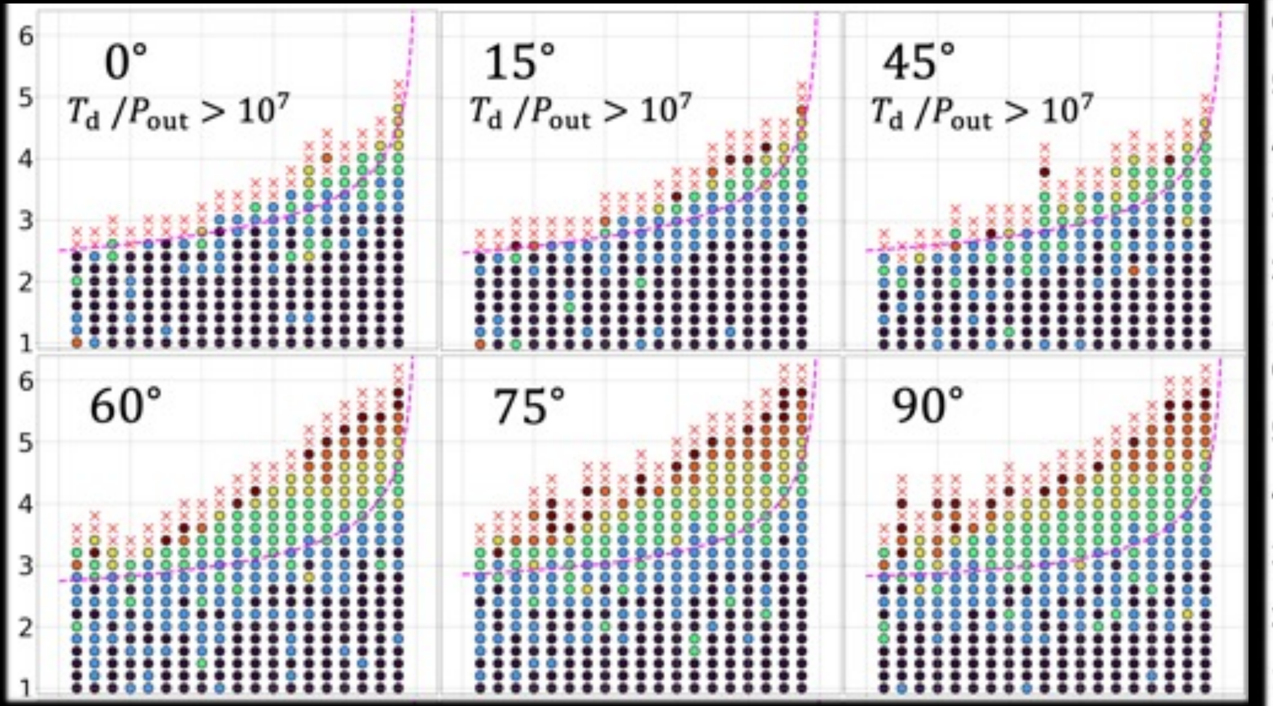
destabilized by the Kozai-Lidov oscillations

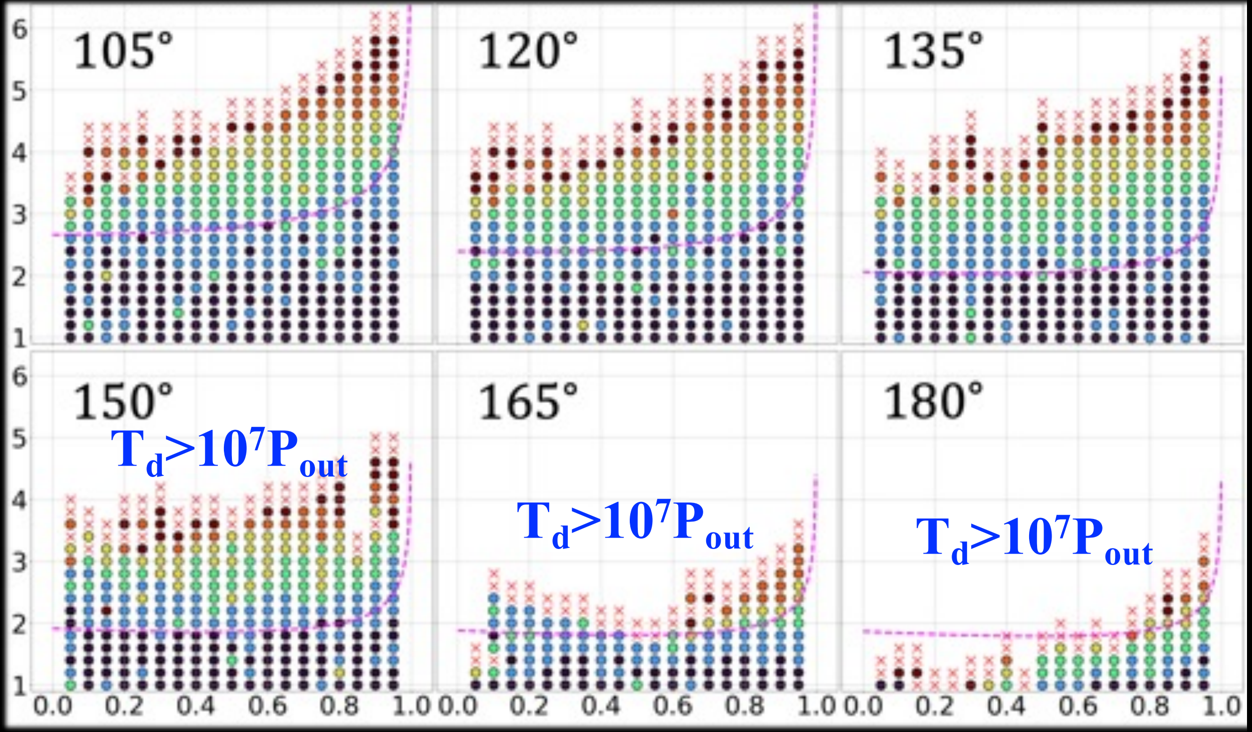
$i_{mut}$ (deg)$>160$

significantly stabilized due to inefficient energy transfer between inner and outer orbits

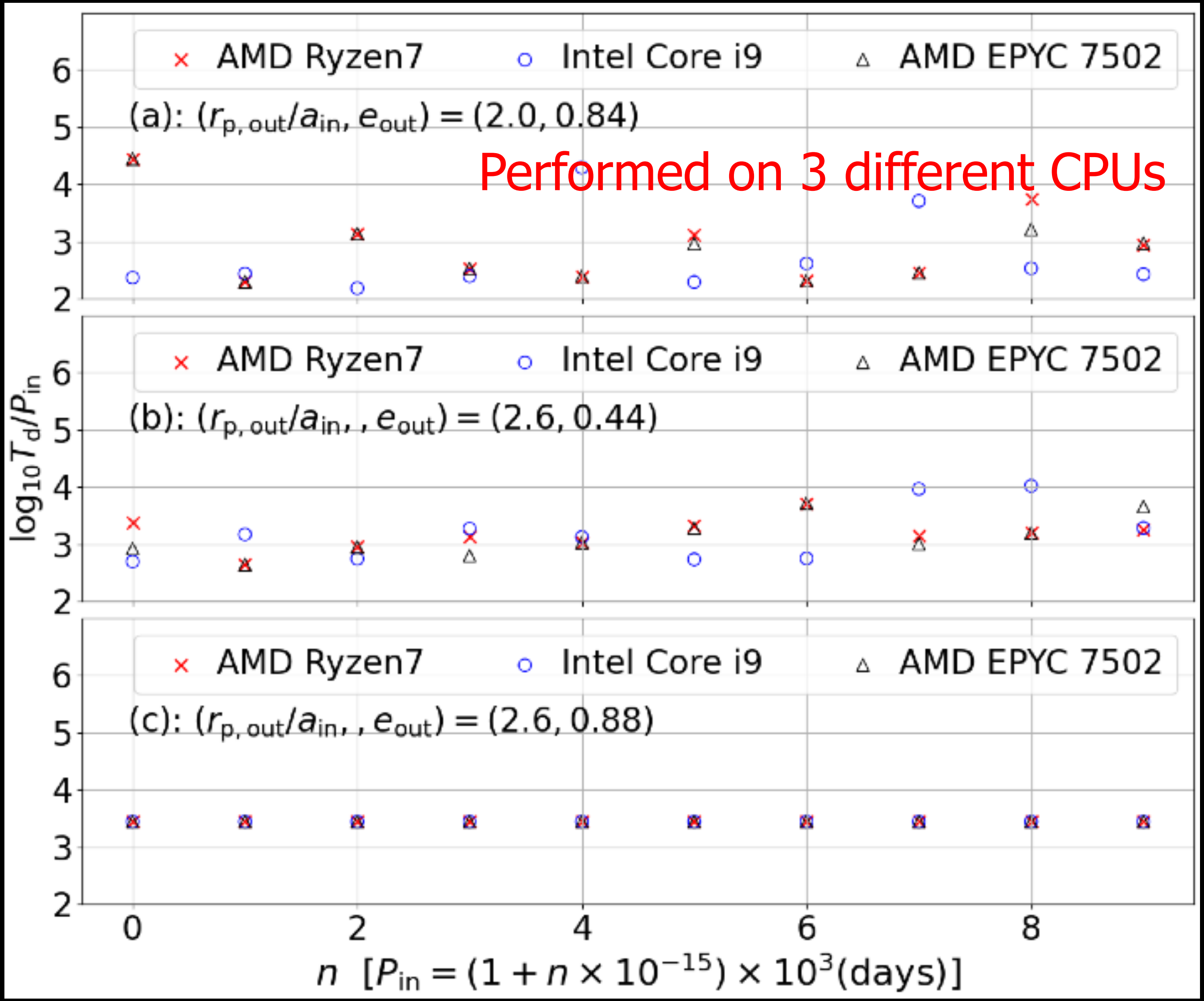



**Very different from the stability boundary by Mardling & Aarseth (2001)**



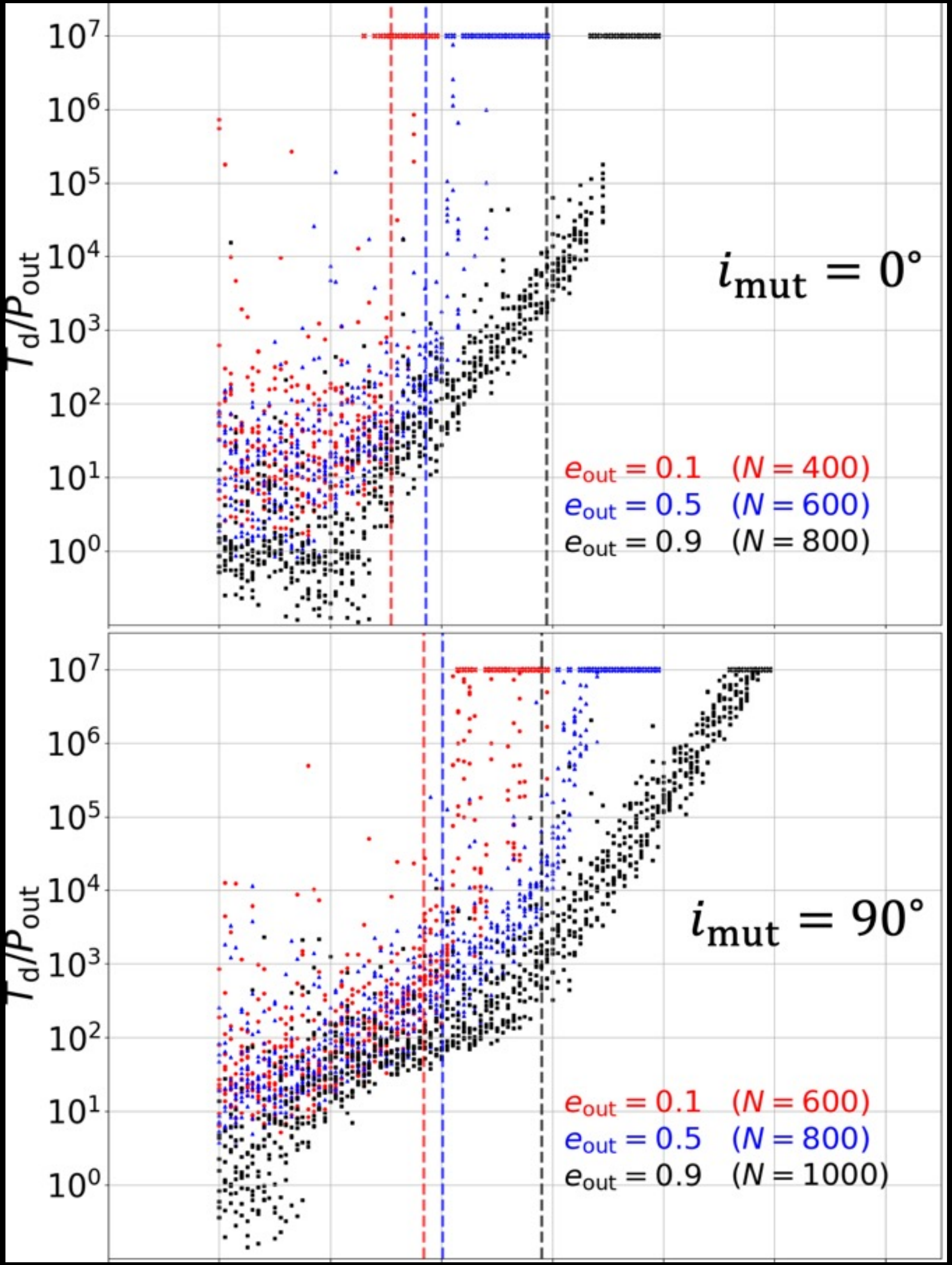
# Chaotic nature of disruption timescale distribution



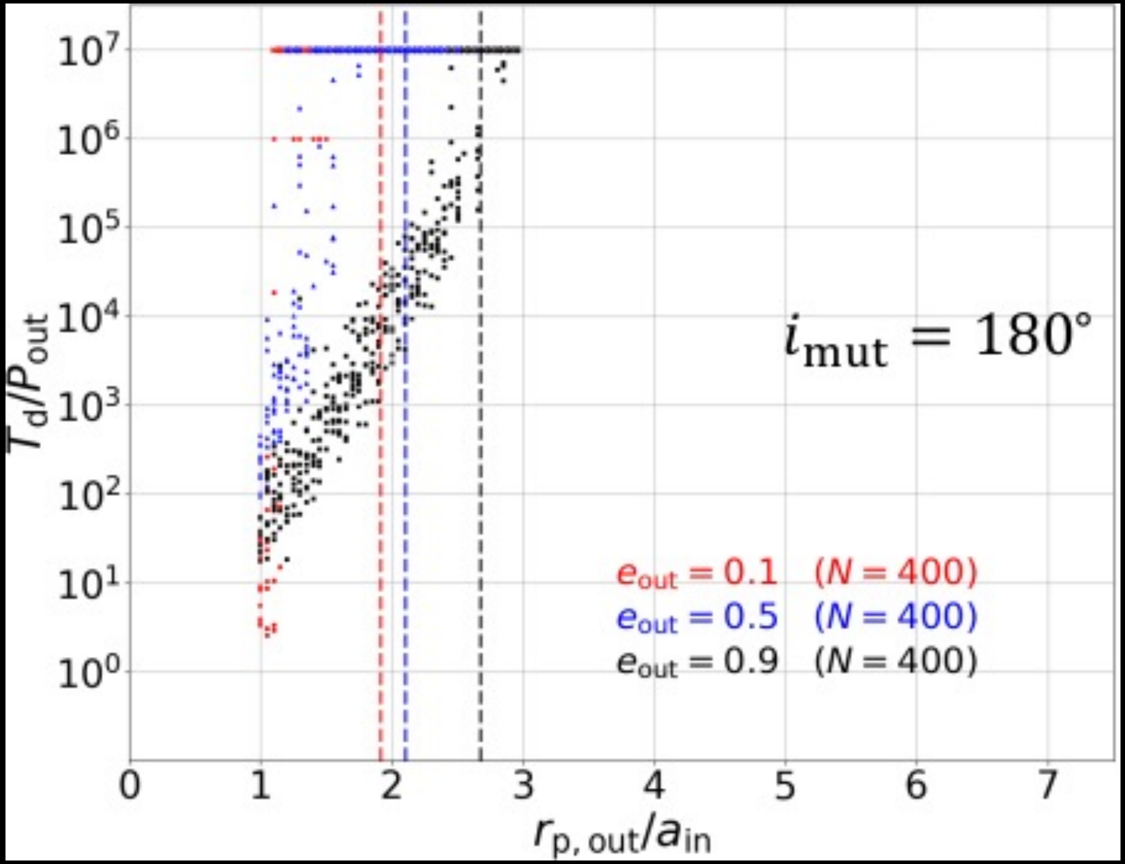

- Tiny difference in the input value of $P_{in}$ leads to one or two order-of-magnitude difference of disruption timescales
- Initial phase difference of the three bodies also leads to one or two order-of-magnitude difference of disruption timescales
- We do not know why one or two order-of-magnitude …

Hayashi, Trani & YS
arXiv:2207.12672
ApJ 939(2022)81

# $T_d/P_{out}$ ($r_{p,out}/a_{in}$; $i_{mut}$, $e_{out}$) with random initial phases




- Lyapunov stability boundaries (Vynatheya et al. 2022) are plotted in dashed lines for reference



# 6 Summary

# Dynamical stability of triple systems

- Lagrange vs. Lyapunov stability for triple systems
  - Conventional criteria correspond to Lyapunov stability
  - Lagrange stability is more relevant in considering the fate of astronomical triples, i.e., disruption timescale
- We derive triple disruption timescales as a function of orbital parameters (within intrinsic variation of one or two order-of-magnitudes due to the chaotic dynamics of triples)
- Strong dependence on the mutual inclination
  - Strongly misaligned systems ($60^\circ < i_{mut} < 150^\circ$) are destabilized due to the Kozai-Lidov oscillations over longer timescales
  - Coplanar retrograde triples are significantly stabilized