Toward detection of gravitational waves by pulsar timing arrays

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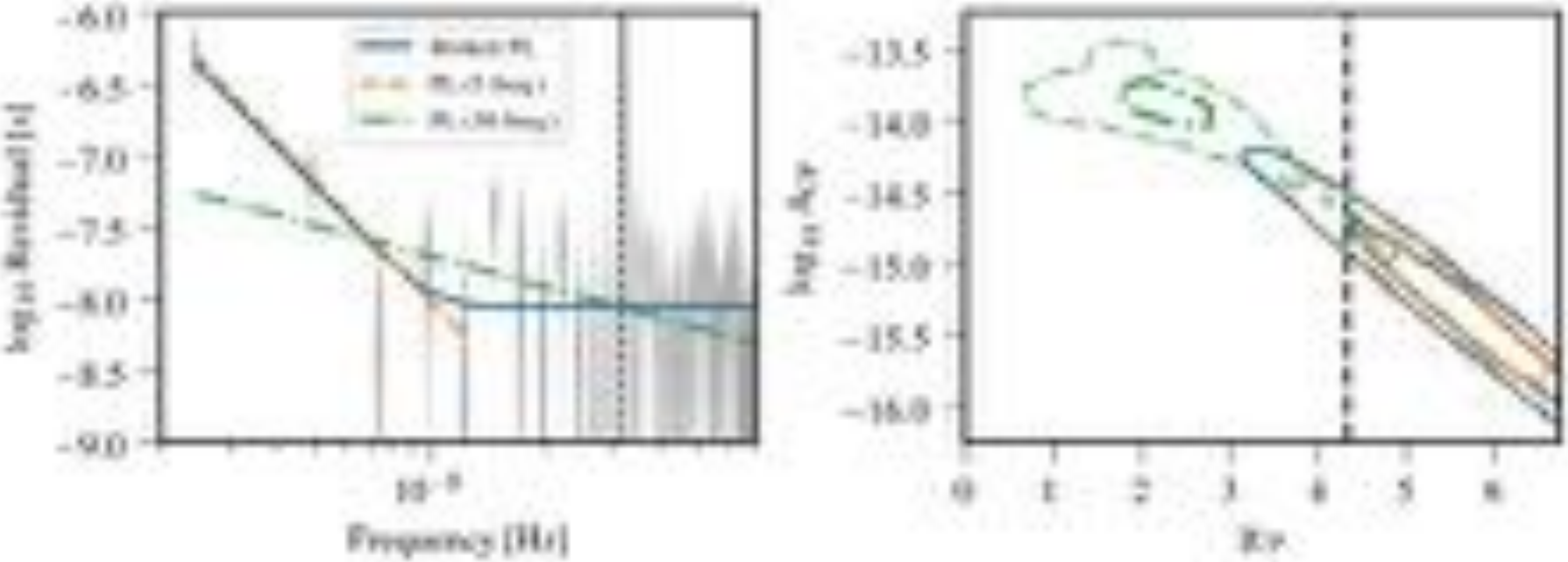

Figure 1. Posteriors for a common-spectrum process in NG12, as recovered with four models: free spectrum (gray violin plots in left panel), broken power law (solid blue lines and contours), five-frequency power law (dashed orange lines and contours), and 30-frequency power law (dot-dashed green lines and contours). In the left panel, the violin plots show marginalized posteriors of the equivalent amplitude of the sine-cosine Fourier pair at the frequencies on the horizontal axis; the lines show the maximum likelihood power laws in the left panel, and the 1-σ (thicker) and 2-σ posterior contours for amplitude and spectral slope in the right panel. The dotted vertical line in the left panel sits at $f_{yr} = 1\,\mathrm{yr}^{-1}$, where PTA sensitivity is reduced by the fitting of pulsar timing-model parameters; the corresponding free-spectrum amplitude posterior is unconstrained. The dashed vertical line in the right panel sits at $\gamma = 13/3$, the expected value for a GWB produced by a population of inspiraling SMBHBs. For both the broken power law and five-frequency power law models, the amplitude (A_{CP}) posterior shown on the right is extrapolated from the lowest frequencies to the reference frequency f_{yr}. We observe that the slope and amplitude of the 30-frequency power law are driven by higher-frequency noise, whereas the five-frequency power law recovers the low-frequency GWB-like slope of the free spectrum and broken power law.

References:

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§ 1 GW effect on the timing of a single pulsar

As usual we work in the TT gauge

$$ds^2 = -dt^2 + [\delta_{ij} + h_{ij}^{TT}(t, x)]dx^i dx^j$$

In this gauge, GW does not shift the position of a star initially at rest.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

take $\frac{dx^i}{d\tau} = 0$ at $\tau = 0$, Then $\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\Gamma^i_{00}\left.\left(\frac{dx^0}{d\tau}\right)^2\right|_{\tau=0} = 0$

$\because$) $\Gamma^i_{00} = \frac{1}{2}(\partial_i h_{0i} - \partial_i h_{00}) = 0$

Suppose that a pulsar exists along the x direction.

x=0

$$dx = -\frac{dt}{\sqrt{1+h^{TT}_{xx}[t,\mathbf{x}(t)]}} \simeq -\left\{1-\frac{1}{2}h_{xx}[t,\mathbf{x}(t)]\right\}dt$$

Distance to the a-th pulsar

$$\underset{\substack{\uparrow \\ \text{a-th pulsar}}}{d_a} = t_{obs} - t_{em} - \frac{1}{2}\int_{t_{em}}^{t_{obs}} dt'\, h^{TT}_{xx}[t', \underbrace{\mathbf{x}(t')}_{\text{0th order photon trajectory}}]$$

$$\mathbf{x}(t') = (t_{obs} - t')\hat{\mathbf{n}}_a$$

For general direction

$$t_{obs} = t_{em} + d_a + \frac{n_a^i n_a^j}{2}\int_{t_{em}}^{t_{em}+d_a} dt'\, h^{TT}_{ij}[t', (t_{em}+d_a-t')\hat{\mathbf{n}}_a]$$

$\mathbf{n}_a$ is a normal vector from the observer to a-th pulsar.

One cycle (Ta) later,

$$t'_{obs} = t_{em} + T_a + d_a + \frac{n_a^i n_a^j}{2} \underbrace{\int_{t_{em}+T_a}^{t_{em}+T_a+d_a} dt'' \, h_{ij}^{TT}\left[t'', (t_{em}+T_a+d_a-t'')\hat{n}_a\right]}_{= \int_{t_{em}}^{t_{em}+d_a} dt' \, h_{ij}^{TT}\left[t'+T_a, (t_{em}+d_a-t')\hat{n}_a\right]}$$

Let $t'_{obs} - t_{obs} \equiv T_a + \Delta T_a$, then

$$\Delta T_a = \frac{n_a^i n_a^j}{2} \int_{t_{em}}^{t_{em}+d_a} dt' \left\{ h_{ij}^{TT}\left[t'+T_a, \vec{x}_0(t')\right] - h_{ij}^{TT}\left[t', \vec{x}_0(t')\right] \right\}$$

$$\vec{x}_0(t') = (t_{em} + d_a - t')\hat{n}_a$$

$$\{\cdots\} \cong \frac{\partial h_{ij}^{TT}}{\partial t'}(t', \vec{x})\Big|_{\vec{x}=\vec{x}_0(t')} T_a$$

$$\frac{\Delta T_a}{T_a} = \frac{1}{2} n_a^i n_a^j \int_{t_{em}}^{t_{em}+d_a} dt' \frac{\partial}{\partial t'} h_{ij}^{TT}(t', \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0(t')}$$

only

Let $h_{ij}^{TT}(t, \mathbf{x}) = A_{ij}(\hat{n}) \cos[\omega_{gw}(t - \hat{n}\cdot\mathbf{x})]$

$$\hat{n}^i A_{ij} = 0 \quad (\because TT)$$

$\hat{\mathbf{n}}$ is a normal vector along which GW propagates

Then

$$\frac{\Delta T_a}{T_a} = -\frac{1}{2} \frac{n_a^i n_a^j A_{ij}}{1 + \hat{n}\cdot\hat{n}_a} \left\{ \cos(\omega_{gw} t_{obs}) - \cos[\omega_{gw} t_{em} - \omega_{gw} \tau_a \hat{n}\cdot\hat{n}_a] \right\}$$

$\mathbf{x} \supset \hat{n}\cdot\hat{n}_a t'$

$$\tau_a \equiv t_{obs} - t_{em}$$

Conventional parameter (redshift)

$$z_a(t) \equiv \left.\frac{\nu_0 - \nu(t)}{\nu_0}\right|_a = -\frac{\Delta\nu_a}{\nu_a} = \frac{\Delta T_a}{T_a}$$

$$z_a(t) = \frac{n_a^i n_a^j}{2(1+\hat{\Omega}\cdot\hat{n}_a)}\left[h_{ij}^{TT}(t,\mathbf{0}) - h_{ij}^{TT}(t-\tau_a, \mathbf{x}_a)\right]$$

$\mathbf{x}_a = d_a \hat{n}_a$ is pulsar's position.

Timing residual (breaks translation invariance of time)

$$R_a(t) \equiv \int_0^t dt' \, z_a(t')$$

Choose a reference frame so that GW propagates along the z direction.

$$\hat{n} = (0, 0, 1) \qquad h_{ij}^{TT}(t-z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_\times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Direction of the a-th pulsar $\hat{n}_a = (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a)$

One easily finds

$$n_a^i n_a^j h_{ij}^{TT} = \sin^2\theta_a (\cos^2\phi_a - \sin^2\phi_a) h_+ + \sin^2\theta_a \, 2\cos\phi_a \sin\phi_a \, h_\times$$

$$= \sin^2\theta_a (h_+ \cos 2\phi_a + h_\times \sin 2\phi_a)$$

so that

$$Z_a(t) = \frac{\sin^2\theta_a}{2(1+\cos\theta_a)} \Big\{ \cos 2\phi_a \left[h_+(t) - h_+(t - \tau_a - \tau_a \cos\theta_a) \right] + \sin 2\phi_a \left[h_\times(t) - h_\times(t - \tau_a - \tau_a \cos\theta_a) \right] \Big\}$$

There is no singularity at cosθ=-1.

Application to (isotropic) stochastic background

$$h_{ij}(t, \mathbf{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int d\Omega_{\hat{n}}\, \tilde{h}_A(f, \hat{n})\, e^A_{ij}(\hat{n})\, e^{-2\pi i f(t - \hat{n}\cdot\mathbf{x})}$$

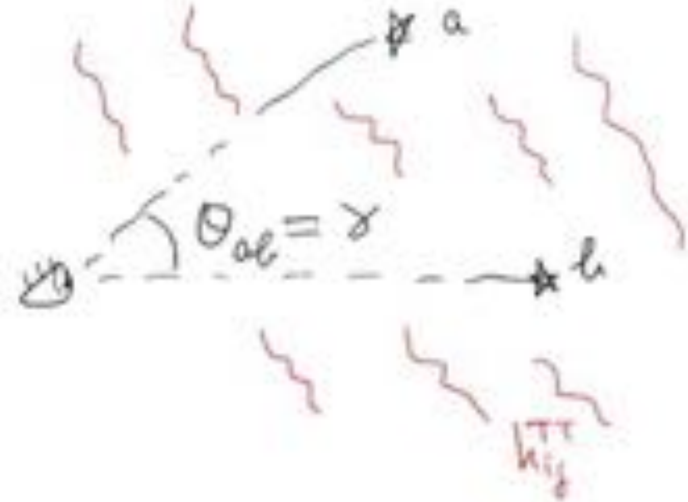
$$z_a(t) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int d\Omega_{\hat{n}}\, \tilde{h}_A(f, \hat{n})\, F^A_a(\hat{n})\, e^{-2\pi i f t}\left[1 - e^{2\pi i f \tau_a (1 + \hat{n}\cdot\hat{n}_a)}\right]$$

$$F^A_a(\hat{n}) = \frac{n^i_a n^j_a e^A_{ij}(\hat{n})}{2(1 + \hat{n}\cdot\hat{n}_a)}$$

Power spectrum

$$\left\langle \tilde{h}^*_A(f, \hat{n})\, \tilde{h}_{A'}(f', \hat{n}') \right\rangle = \delta(f, f')\, \frac{\delta^2(\hat{n}, \hat{n}')}{4\pi}\, \delta_{AA'} \times \frac{1}{2} S_h(f)$$

§ 2 GW effect on the timing of a pair of pulsars



Equal time two-point function

$$\langle z_a(t) z_b(t) \rangle = \frac{1}{2}\int_{-\infty}^{\infty} df\, S_h(f) \int \frac{d\Omega_{\hat{n}}}{4\pi} \mathcal{K}_{ab}(f, \hat{n}) \sum_{A=+\times} F_a^A(\hat{n}) F_b^A(\hat{n})$$

$$\mathcal{K}_{ab}(f, \hat{n}) \equiv \left[1 - e^{-2\pi i f \tau_a (1+\hat{n}\cdot\hat{n}_a)}\right]\left[1 - e^{2\pi i f \tau_b (1+\hat{n}\cdot\hat{n}_b)}\right]$$

Polarization tensors of GW $e^{A}_{ij}(\hat{n})$

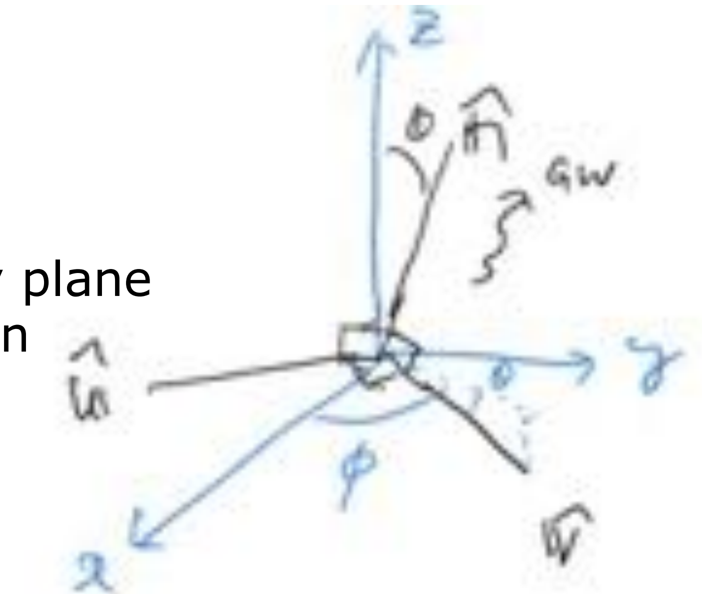
$$\hat{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\hat{u} = (\sin\phi,\ -\cos\phi,\ 0)$$

a vector on xy plane orthogonal to n

$$\hat{v} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$$

a vector orthogonal to both n and u



$\hat{n}, \hat{u}, \hat{v}$ span orthnormal basis.

$$e^{+}_{ij}(\hat{n}) = \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j$$

$$e^{\times}_{ij}(\hat{n}) = \hat{u}_i\hat{v}_j + \hat{u}_j\hat{v}_i$$

Recall that we find $h^{TT}_{ij}(t-z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_\times & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $\hat{n} = (0,0,1)$

$$F_a^+(\hat{n}) = \frac{\hat{n}_a^i \hat{n}_a^j e_{ij}^+(\hat{n})}{2(1+\hat{n}\cdot\hat{n}_a)} = \frac{\hat{n}_a^i \hat{n}_a^j(\hat{u}_i\otimes\hat{u}_j - \hat{v}_i\otimes\hat{v}_j)}{2(1+\hat{n}\cdot\hat{n}_a)} = \frac{(\hat{n}_a\cdot\hat{u})^2-(\hat{n}_a\cdot\hat{v})^2}{2(1+\hat{n}\cdot\hat{n}_a)}$$

$$F_a^\times(\hat{n}) = \frac{(\hat{n}_a\cdot\hat{u})(\hat{n}_a\cdot\hat{v})}{1+\hat{n}\cdot\hat{n}_a}$$

To calculate the angular integral of $\langle z_a(t) z_b(t)\rangle = \frac{1}{2}\int_{-\infty}^{\infty} df\, S_h(f) \boxed{\int \frac{d\Omega_{\hat{n}}}{4\pi} \mathcal{R}_{ab}(f,\hat{n}) \sum_{A=+\times} F_a^A(\hat{n}) F_b^A(\hat{n})}$

only the relative angle $\theta_{ab} = \gamma$ between a & b matters.

So one may choose $\hat{n}_a = (0,0,1)$ and $\hat{n}_b = (\sin\gamma,\ 0,\ \cos\gamma)$

$$F_a^+(\hat{n}) = \frac{-\sin^2\theta}{2(1+\cos\theta)} \left(= \frac{1}{2}(\cos\theta - 1) \right)$$

$$F_a^\times(\hat{n}) = \frac{-\cos\theta\sin\theta}{1+\cos\theta}$$

$$F_b^+(\hat{n}) = \frac{(\sin\phi\sin\gamma)^2 - (\sin\gamma\cos\theta\cos\phi - \cos\gamma\sin\theta)^2}{2(1+\sin\gamma\sin\theta\cos\phi + \cos\gamma\cos\theta)}$$

$$F_b^\times(\hat{n}) = \frac{\sin\gamma\sin\phi(\sin\gamma\cos\theta\cos\phi - \cos\gamma\sin\theta)}{1+\sin\gamma\sin\theta\cos\phi+\cos\gamma\cos\theta}$$

Calculation of angular integral is tedious but possible, to yield

$$\sum_{A=+\times} \int \frac{d\Omega_{\hat{n}}}{4\pi} F_a^A(\hat{n}) F_b^A(\hat{n}) = x_{ab} \log x_{ab} - \frac{1}{6} x_{ab} + \frac{1}{3} \equiv C(\theta_{ab})$$

$$x_{ab} \equiv \frac{1}{2}(1 - \cos\theta_{ab})$$

This angular function was first obtained by Hellings and Downs, and its derivation is presented in Appendix C of Anholm et al (2009).

$$\langle z_a(t) z_b(t) \rangle = \underbrace{C(\theta_{ab})}_{} \underbrace{\int_0^\infty df\, S_h(f)}_{}$$

separable

This angular dependence is most important for the detection of stochastic gravitational waves in terms of pulsar timing.

In terms of the timing residual

$$R_a(t) = \int_0^t z_a(t')\,dt' = \sum_A \int_{-\infty}^{\infty} df \int d\Omega_{\hat{n}}\, \tilde{h}_A(f,\hat{n}) F_a^A(\hat{n}) \frac{e^{-2\pi i f} - 1}{-2\pi i f}$$

one can calculate the equal-time correlation function.

$$r_{ab}(t) \equiv \langle R_a(t) R_b(t) \rangle = C(\theta_{ab}) \int_0^{\infty} df \frac{S_h(f)}{(2\pi f)^2} \times \underline{2\left[1 - \cos(2\pi f t)\right]}$$

$$\int_0^{\infty} \to \int_{f_l}^{f_h} df \quad \text{practically}$$

$$2\left[1 - \cos(2\pi f t)\right] \simeq (2\pi f t)^2 \quad \text{for } 2\pi f_h t \ll 1$$

$$\therefore\ r_{ab}(t) \simeq t^2 \langle z_a(t) z_b(t) \rangle$$

Define

$$P_g(f) \equiv \frac{2}{3} \frac{S_h(f)}{(2\pi f)^2}$$

and assume $2\left[1 - \cos(2\pi f t)\right] \simeq 1$

the we find

$$r_{ab} \simeq \zeta(\theta_{ab}) \int_{f_l}^{f_h} df\, P_g(f), \quad \zeta(\theta_{ab}) \equiv \frac{3}{2} C(\theta_{ab})$$

$$r_{al} \simeq \zeta(\theta_{al}) \int_{f_L}^{f_H} df \, P_g(f), \quad \zeta(\theta_{al}) \equiv \frac{3}{2} C(\theta_{al})$$

Define $r_{al} \equiv \int_0^\infty d\log f \, \frac{d r_{al}}{d \log f}$ so that $\frac{d r_{al}(f)}{d \log f} = \zeta(\theta_{al}) \, f \, P_g(f)$

Define the amplitude by $h_c^2(f) \equiv 2f \, S_h(f)$

then

$$\frac{d r_{al}}{d \log f} = \zeta(\theta_{al}) \frac{h_c^2(f)}{12\pi^2 f^2}$$

We often assume a power law strain $h_c(f) = A_* \left(\frac{f}{f_*}\right)^\alpha$

with which the spectrum reads $S_h(f) = \frac{h_c^2(f)}{2f} = \frac{A_*^2}{2f_*}\left(\frac{f}{f_*}\right)^{2\alpha-1}$

Calculation of unequal-time two-point function is also possible (Ref 3).

$$R_a(t) = \int_0^t z_a(t')\,dt' = \sum_A \int_{-\infty}^{\infty} df \int d\Omega_{\hat{n}}\, \tilde{h}_A(f,\hat{n}) F_a^A(\hat{n}) \frac{e^{-2\pi i f} - 1}{-2\pi i f}$$

$$\langle R_a(t) R_b(t')\rangle = C(\theta_{ab}) \int_0^{\infty} df\, \frac{S_h(f)}{(2\pi f)^2} \left(e^{-2\pi i f t} - 1\right)\left(e^{2\pi i f t'} - 1\right)$$

$$= \zeta(\theta_{ab}) \int_{f_L}^{f_H} df \frac{A_*^2}{8\pi^2 f_*^{2\alpha}} f^{2\alpha-3} \left(e^{-2\pi i f(t-t')} - e^{2\pi i f t'} - e^{-2\pi i f t} + 1\right)$$

Exchange t and t′ and take average to make them real-valued.

$$\simeq \zeta(\theta_{ab}) \int_{f_L}^{\infty} \frac{A_*^2}{8\pi^2 f_*^{2\alpha}} f^{-\gamma-2} \cos\left[2\pi(t-t')f\right] df$$

$$\gamma \equiv 1 - 2\alpha$$

$$\langle R_a(t) R_a(t') \rangle \simeq \zeta(\theta_{ab}) \int_{f_L}^{\infty} \frac{A_*^2}{8\pi^2 f_*^{2\alpha}} f^{-\gamma-2} \cos[2\pi(t-t')f] \, df$$

dividing the integral as

$$\int_{f_L}^{\infty} df = \int_0^{\infty} df - \int_0^{f_L} df$$ and using $\cos \tau f = \sum_{k=0}^{\infty} (-)^k \frac{(\tau f)^{2k}}{(2k)!}$ for the latter

$\tau = 2\pi|t-t'|$

we find $$\int_0^{f_L} df \, f^{-\gamma-2} \cos \tau f = \int_0^{f_L} df \sum_{k=0}^{\infty} (-)^k \frac{(f\tau)^{2k}}{(2k)!} f^{-\gamma-2}$$

so that

$$\langle R_a(t) R_a(t') \rangle = \zeta(\theta_{ab}) \frac{A_* f_*^{\gamma}}{(2\pi)^2 f_L^{\gamma+1}} \left\{ \boxed{\Gamma(1-\gamma) \sin\left(-\frac{\pi\gamma}{2}\right) (f_L \tau)^{\gamma+1}} - \sum_{k=0}^{\infty} (-)^k \frac{(f_L \tau)^{2k}}{(2k)! \, (2k-\gamma-1)} \right\}$$

$$\gamma \equiv 1 - 2\alpha$$

§ 3 Statistics

Observables: Sequence of timing residuals $R_a(t_i)$ of a-th pulsar which are collectively denoted by

$a = 1, 2, \cdots N_p$
$i = 1, 2 \cdots$

$$\mathbb{t}^{obs} = \underbrace{\mathbb{t}^{det}}_{\text{deterministic part depending on pulsar parameters } \xi}(\xi_{true}) + \underbrace{\mathbb{n}}_{\text{noises}}$$

$$\underset{\text{prefit}}{\delta \mathbb{t}^{pre}} = \mathbb{t}^{obs} - \mathbb{t}^{det}(\xi_{est}) = \mathbb{t}^{det}(\xi_{true}) - \mathbb{t}^{det}(\xi_{est}) + \mathbb{n}$$

$$= \left.\frac{\partial \mathbb{t}^{det}}{\partial \xi}\right|_{\xi_{true}} \delta\xi + \mathbb{n} \equiv M\delta\xi + \mathbb{n}$$

Assuming that noises are Gaussian distributed,

$$P(\mathbf{n} \mid \Theta) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left[-\frac{1}{2}\,{}^t\mathbf{n}\,\Sigma^{-1}\mathbf{n}\right]$$

parameters

$\Sigma = \langle \mathbf{n}\,{}^t\mathbf{n} \rangle$ covariance matrix of noises (including GW, which acts as a noise for timing data).

$$\Sigma = \begin{pmatrix} N_1 & X_{12} & \cdots & X_{1N_p} \\ X_{21} & N_2 & & \\ \vdots & & \ddots & \\ X_{N_p1} & & & N_{N_p} \end{pmatrix}$$

$N_a = \langle \mathbf{n}_a\,{}^t\mathbf{n}_a \rangle$

for a-th pulsar

$X_{ab} = \langle \mathbf{n}_a\,{}^t\mathbf{n}_b \rangle \quad a \neq b$

represents correlation between pulsars a and b

$$\tau_{ij} = (t_i - t_j)$$

$$N_a = \langle n_a(t_i)\, n_a(t_j) \rangle = \underbrace{\int_0^\infty df\, e^{2\pi i f \tau_{ij}} P_I(f)}_{\text{frequency dep't red noise}} + \underbrace{\sigma^2_{W ij}}_{\text{frequency independent white noise part}} \sim\delta(t_i - t_j)$$

White noise: temporarily uncorrelated, such as instrumental error

Red noise: time correlated noises with excess power at low frequency including GW and intrinsic rotational instability

$$X_{ab} = \langle n_a(t_i)\, n_b(t_j) \rangle = \zeta(\theta_{ab}) \int_0^\infty df\, e^{2\pi i \tau_{ij}} P_g(f)$$ + monopole and dipole terms

GW

$$\langle R_a(t) R_b(t') \rangle$$

monopole term: earth clock
dipole: solar system model (measurement is done on Earth and interpretation is done at the barycenter of the solar system)

$$P(\delta t \mid \Theta, \delta\xi) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left[-\frac{1}{2}\,{}^t\underbrace{(\delta t - M\delta\xi)}_{\text{noise } n}\,\Sigma^{-1}(\delta t - M\delta\xi)\right]$$

Marginalize over $N_{parameter}$ pulsar parameters $\delta\xi$

$$P(\delta t \mid \Theta) = \frac{1}{\sqrt{\det(2\pi\bar{\bar{\Sigma}})}} \exp\left[-\frac{1}{2}\,{}^t\delta t\; G\,({}^tG\Sigma G)^{-1} G\,\delta t\right]$$

$$= \frac{1}{\sqrt{\det(2\pi\bar{\bar{\Sigma}})}} \exp\left[-\frac{1}{2}\,{}^t r\,\bar{\bar{\Sigma}}\, r\right]$$

$$r = {}^tG\,\delta t \qquad {}^tG\Sigma G = \bar{\bar{\Sigma}}$$

$$G = \mathrm{diag}(G_1, G_2, \cdots G_a \cdots G_{N_p})$$

$$G_a : \underbrace{N_{TOA}}_{\text{time of arrival}} \times (N_{TOA} - N_{parameter}) \text{ matrix}$$

$$P_a \equiv {}^t G_a N_a G_a \qquad \Rightarrow \qquad \bar{\bar{\Sigma}} = \begin{pmatrix} P_1 & S_{12} & \cdots & S_{1N_p} \\ S_{21} & P_2 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ S_{N_p 1} & \cdots & & P_{N_p} \end{pmatrix}$$

$$S_{ab} \equiv {}^t G_a X_{ab} G_b$$

Likelihood function

$$\log \underline{P}(\boldsymbol{r} \mid \Theta) = -\frac{1}{2} {}^t\boldsymbol{r}\, \bar{\bar{\Sigma}}^{-1} \boldsymbol{r} - \frac{1}{2} \ln\left[\det(2\pi \bar{\bar{\Sigma}})\right]$$

Let $\bar{\bar{\Sigma}} \equiv P + \epsilon S$ with epsilon being a small expansion parameter

which turns out to be the square amplitude of GW. $\epsilon = A_{gw}^2$

$$\bar{\bar{\Sigma}}^{-1} = (P + \epsilon S)^{-1} = P^{-1} - \epsilon P^{-1} S P^{-1} + O(\epsilon^2)$$

$$\because) \quad \bar{\bar{\Sigma}}\,\bar{\bar{\Sigma}}^{-1} = (P + \epsilon S)(P^{-1} - \epsilon P^{-1} S P^{-1})$$

$$= PP^{-1} + \epsilon S P^{-1} - \epsilon P P^{-1} S P^{-1} = PP^{-1} + O(\epsilon^2) = \mathbb{1}$$

$$\log P(\boldsymbol{r} \mid \Theta) = -\frac{1}{2} {}^t\boldsymbol{r}\, \bar{\bar{\Sigma}}^{-1} \boldsymbol{r} - \frac{1}{2} \ln\left[\det(2\pi \bar{\bar{\Sigma}})\right]$$

$$\log P(\boldsymbol{r} \mid \Theta) \cong -\frac{1}{2}\left[{}^t\boldsymbol{r}_a P_a^{-1} \boldsymbol{r}_a + \mathrm{tr} \log P_a - \epsilon\, {}^t\boldsymbol{r}_a P_a^{-1} S_{ab} P_b^{-1} \boldsymbol{r}_b \right]$$

Log likelihood ratio between the cases with GW and without

$$\log \Lambda = \log P(\boldsymbol{r} \mid \Theta_{GW}) - \log P(\boldsymbol{r} \mid \Theta_{noise})$$

$$= \frac{1}{2} A_{gw}^2\, {}^t\boldsymbol{r}_a P_a^{-1} \tilde{S}_{ab} P_b^{-1} \boldsymbol{r}_b$$

where we have defined

$$S_{ab} = \langle \boldsymbol{r}_a\, {}^t\boldsymbol{r}_b \rangle \equiv A_{gw}^2 \tilde{S}_{ab}$$

Optimal statistic to calculate the amplitude of GW

$$\hat{A}^2 = \frac{{}^t\mathbb{r}_a P_a^{-1} \tilde{S}_{ab} P_b^{-1} \mathbb{r}_b}{\mathrm{tr}[P_c^{-1} \tilde{S}_{cd} P_d^{-1} \tilde{S}_{dc}]}$$

so that $\langle \hat{A}^2 \rangle = A_{gw}^2$

$$\because) \quad \langle {}^t\mathbb{r}_a P_a^{-1} \tilde{S}_{ab} P_b^{-1} \mathbb{r}_b \rangle = \mathrm{tr}[P_a^{-1} \tilde{S}_{ab} P_b^{-1} S_{ba}]$$

$$= A_{gw}^2 \, \mathrm{tr}[P_a^{-1} \tilde{S}_{ab} P_b^{-1} \tilde{S}_{ba}]$$

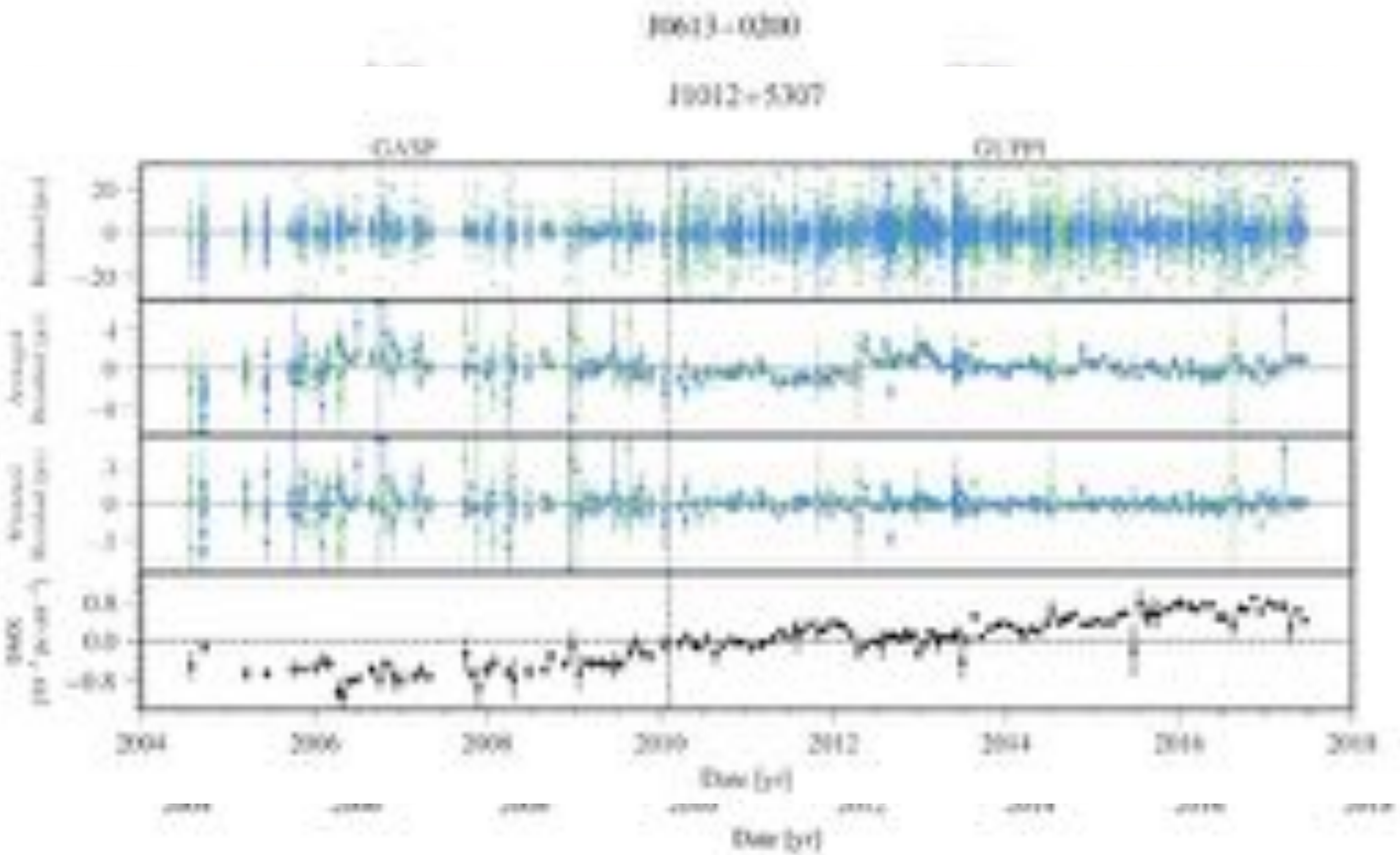
If signal is weak, the standard deviation is given by

$$\sigma_0 = \left(\mathrm{tr}[P_a^{-1} \tilde{S}_{ab} P_b^{-1} \tilde{S}_{ba}]\right)^{1/2}$$

so the SN ratio reads

$$\hat{\rho} = \frac{\hat{A}^2}{\sigma_0} = \frac{{}^t\mathbb{r}_a P_a^{-1} \tilde{S}_{ab} P_b^{-1} \mathbb{r}_b}{\left(\mathrm{tr}[P_a^{-1} \tilde{S}_{ab} P_b^{-1} \tilde{S}_{ba}]\right)^{1/2}}$$

§4 NANOGrav 12.5year data analysis



Solar system ephemeris(SSE)（天体暦）　JPL’s DE421—DE436

Errors in Jupiter’s orbit induces the largest systematic error to GW, whose uncertainty is about 50km.

A_{CP} amplitude of common power spectrum

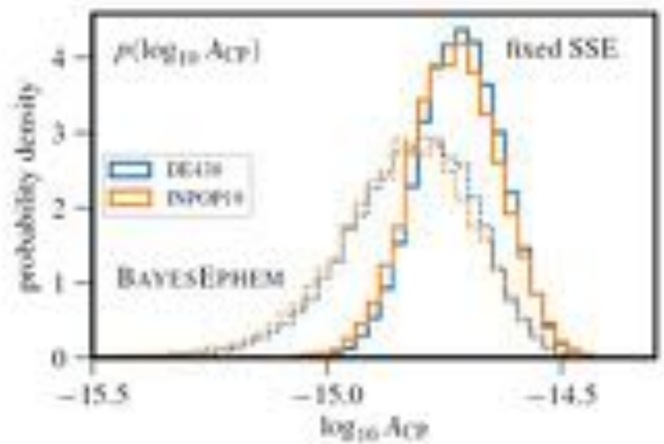


Figure 2. Bayesian posteriors for the ($f_{yr} = 1 yr^{-1}$) amplitude A_{CP} of a common-spectrum process, modeled as a $\gamma = 13/3$ power law using only the lowest five component frequencies. The posteriors are computed for the NANOGrav 12.5-year data set using individual ephemerides (solid lines), and BAYESEPHEM (dotted). Unlike similar analyses in NG11gwb and Vallisneri et al. (2020), these posteriors, even those using BAYESEPHEM imply a strong preference for a common-spectrum process. Results are consistent for both recent SSEs (DE438 and INPOP19a) updated with Jupiter data from the mission Juno. SSE corrections remain partially entangled with A_{CP} Thus when BAYESEPHEM is applied, the distributions broaden toward lower amplitudes shifting the peak of the distribution by $\sim 20\%$.

Baysian analysis (d: observed data, Mi, Mj: models)

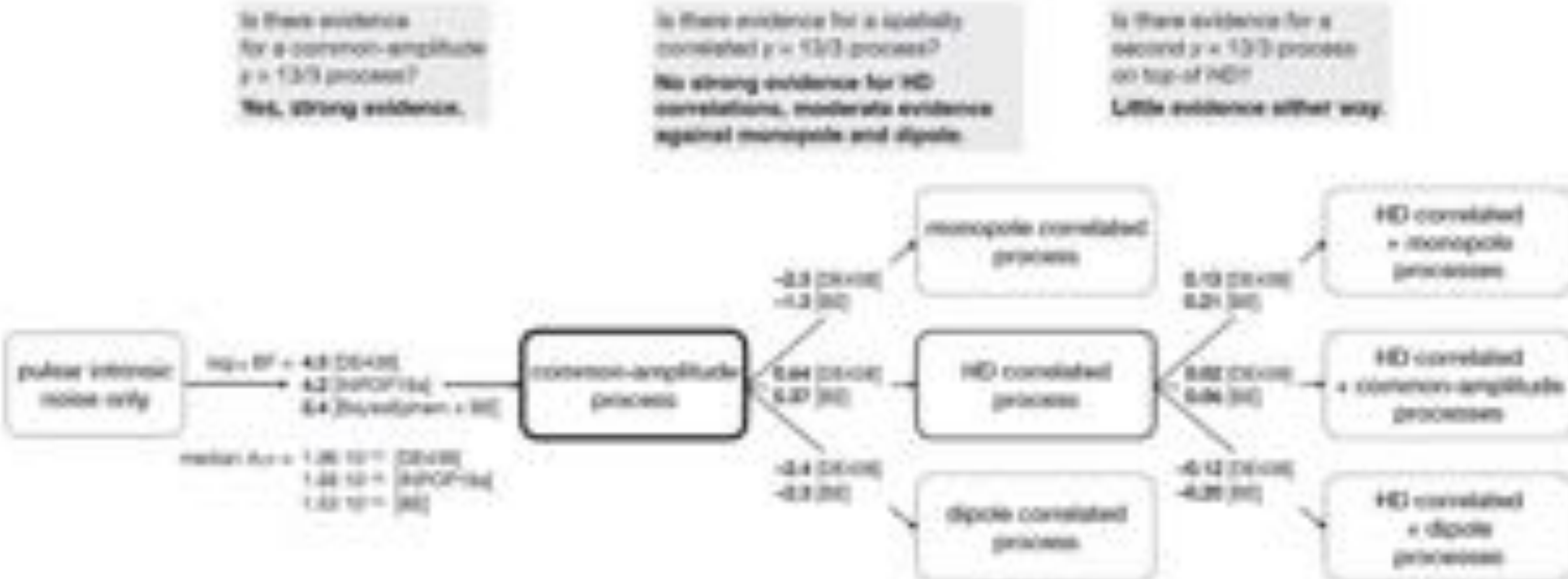
P(Mi|d): The probability of the model Mi being correct for a given set of observational data d. (This is what we want to obtain.)

$$\frac{P(Mi|d)}{P(Mj|d)} = \frac{P(d|Mi)}{P(d|Mj)}\frac{P(Mi)}{P(Mj)} = Bij \boxed{\frac{P(Mi)}{P(Mj)}}$$

prior probability ratio
(usually taken flat)

Bayes factor Bij : Relative probability of Mi to Mj

Bij=1-3	weak	$\log_{10}$Bij=	0-0.48	weak
3-20	definite		0.48-1.3	definite
20-150	strong		1.3-2.2	strong
>150	very strong		>2.2	very strong



$\log_{10}$Bij=	0-0.48	weak
	0.48-1.3	definite
	1.3-2.2	strong
	>2.2	very strong

Table 2. Bayesian model-comparison scores

ephemeris	uncorr. process vs. noise-only	dipole (vs. uncorrelated process)	mono. (vs. uncorrelated process)	HD (vs. uncorrelated process)	HD+dip. (vs. HD correlated process)	HD+mono. (vs. HD correlated process)	HD+uncorr. (vs. HD correlated process)
DE438	4.5(9)	−2.4(2)	−2.3(2)	0.64(1)	−0.116(4)	0.126(4)	0.0164(1)
BayesEphem	2.4(2)	−2.3(2)	−1.3(1)	0.371(5)	−0.199(5)	0.217(6)	0.0621(4)

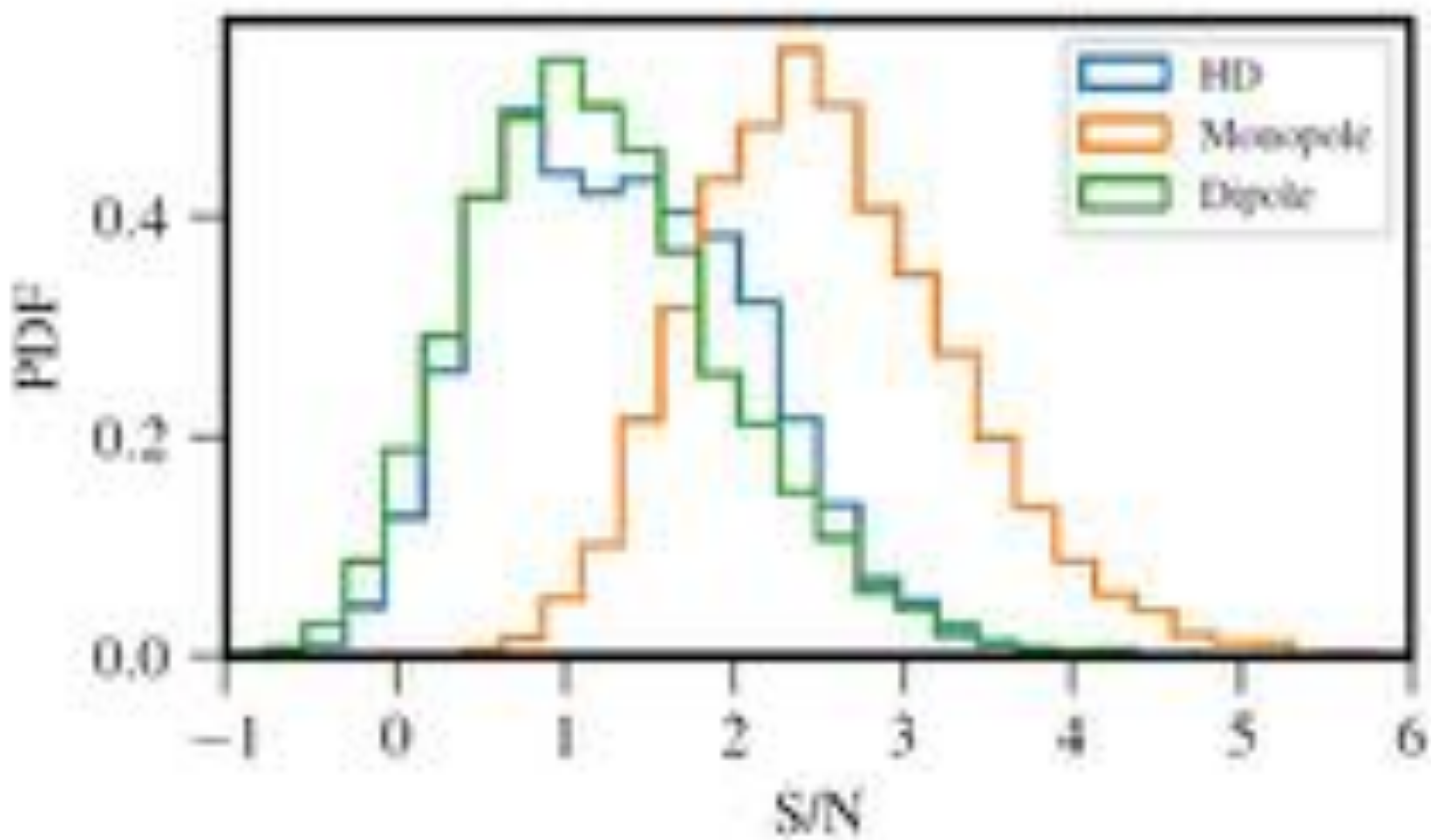
TE—The $\log_{10}$ Bayes factors between pairs of models from Table 1 are also visualized in Figure 3. All common-spectrum wer-law processes are modeled with fixed spectral index $\gamma = 13/3$ and with the lowest five frequency components. The digit he parentheses gives the uncertainty on the last quoted digit.

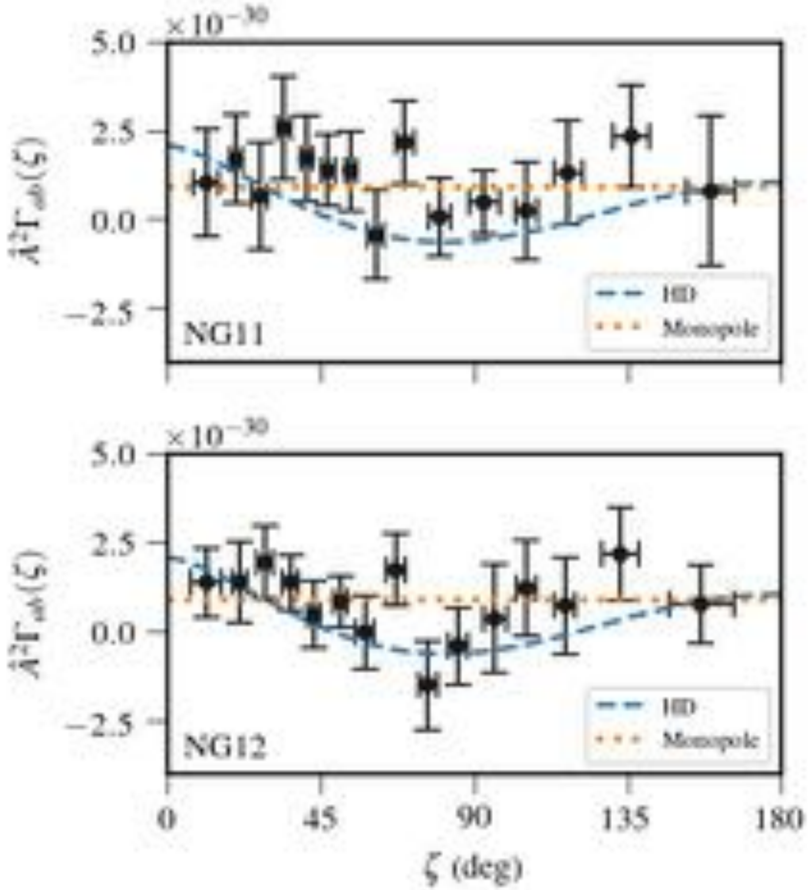
$$\hat{A}^2 = \frac{{}^t \boldsymbol{r}_a P_a^{-1} \tilde{S}_{ab} P_b^{-1} \boldsymbol{r}_b}{\mathrm{tr}\left[P_c^{-1} \tilde{S}_{cd} P_d^{-1} \tilde{S}_{dc} \right]} \qquad \hat{\rho} = \frac{\hat{A}^2}{\sigma_0} = \frac{{}^t \boldsymbol{r}_a P_a^{-1} \hat{S}_{ab} P_b^{-1} \boldsymbol{r}_b}{\left(\mathrm{tr}\left[P_a^{-1} \tilde{S}_{ab} P_b^{-1} \tilde{S}_{ba} \right] \right)^{1/2}}$$

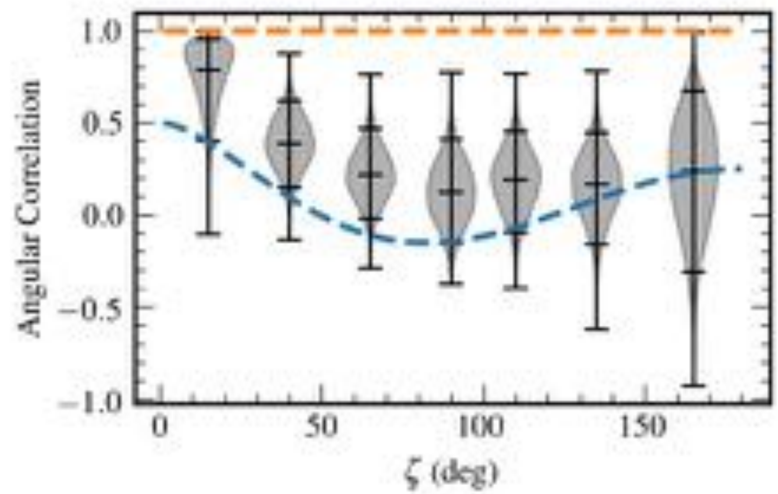
	fixed noise		noise marginalized	
correlation	$\hat{A}^2$	S/N	mean $\hat{A}^2$	mean S/N
HD	4×10^{-30}	2.8	$2(1) \times 10^{-30}$	1.3(8)
monopole	9×10^{-31}	3.4	$8(3) \times 10^{-31}$	2.6(8)
dipole	9×10^{-31}	2.4	$5(3) \times 10^{-31}$	1.2(8)

NOTE—The optimal statistic, $\hat{A}^2$, and corresponding S/N are computed from the 12.5-year data set for a HD, monopolar, and dipolar correlated common-process modeled as a power-law with fixed spectral index, $\gamma = 13/3$, using the five lowest frequency components. We show fixed intrinsic red-noise and noise-marginalized values. All are computed with fixed ephemeris DE438.

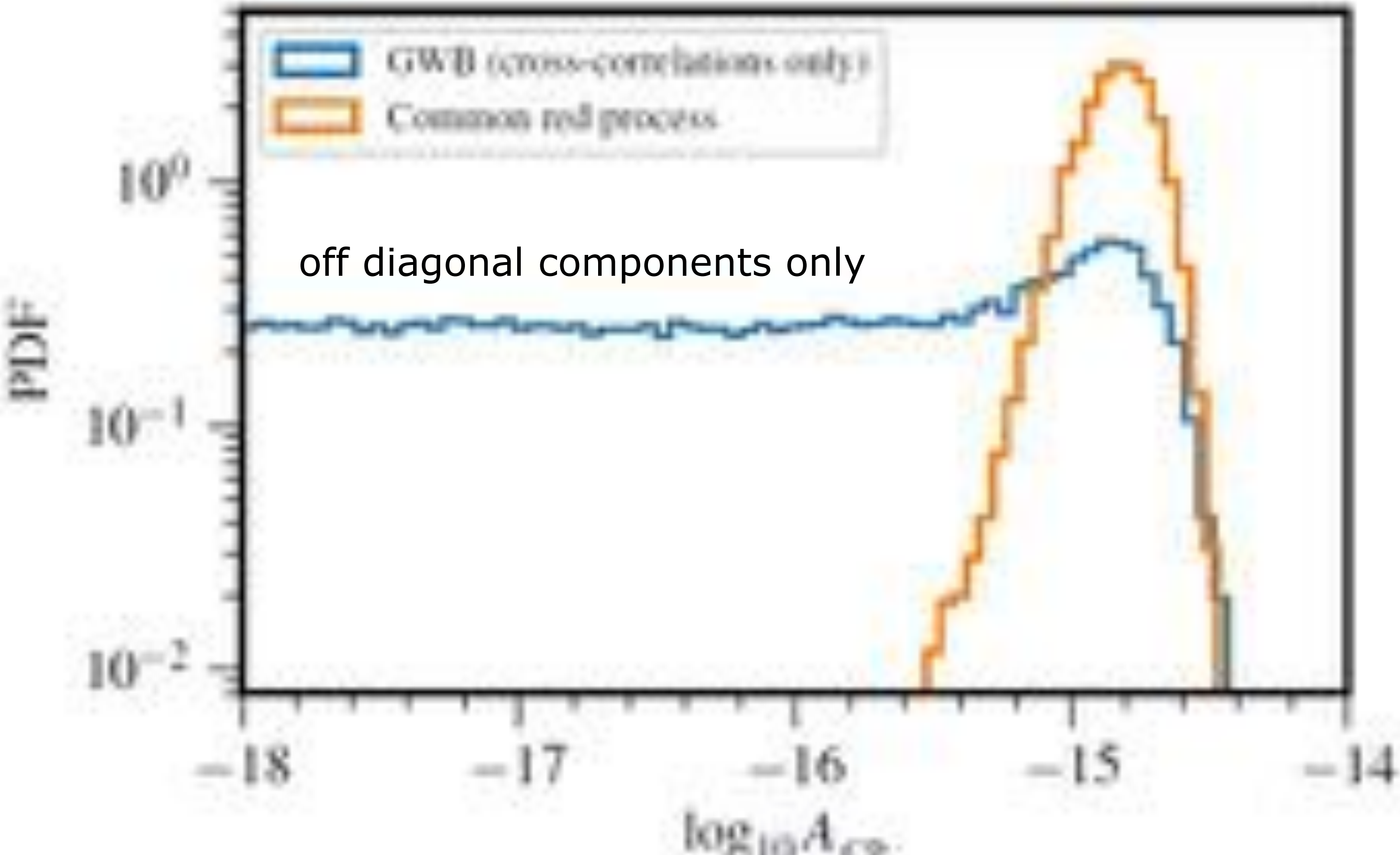
$$\hat{\rho} = \frac{\hat{A}^2}{\sigma_0} = \frac{{}^t\kappa_a P_a^{-1} \hat{S}_{ab} P_b^{-1} \kappa_b}{\left(\mathrm{tr}\left[P_a^{-1} \tilde{S}_{ab} P_b^{-1} \tilde{S}_{ba}\right]\right)^{1/2}}$$







Bayesian reconstruction


GWB (cross-correlations only)
Common red process
off diagonal components only
PDF
10^{0}
10^{-1}
10^{-2}
−18
−17
−16
−15
−14
$\log_{10} A_{\mathrm{CP}}$

Estimating the probability to obtain the observed SN ratio by chance

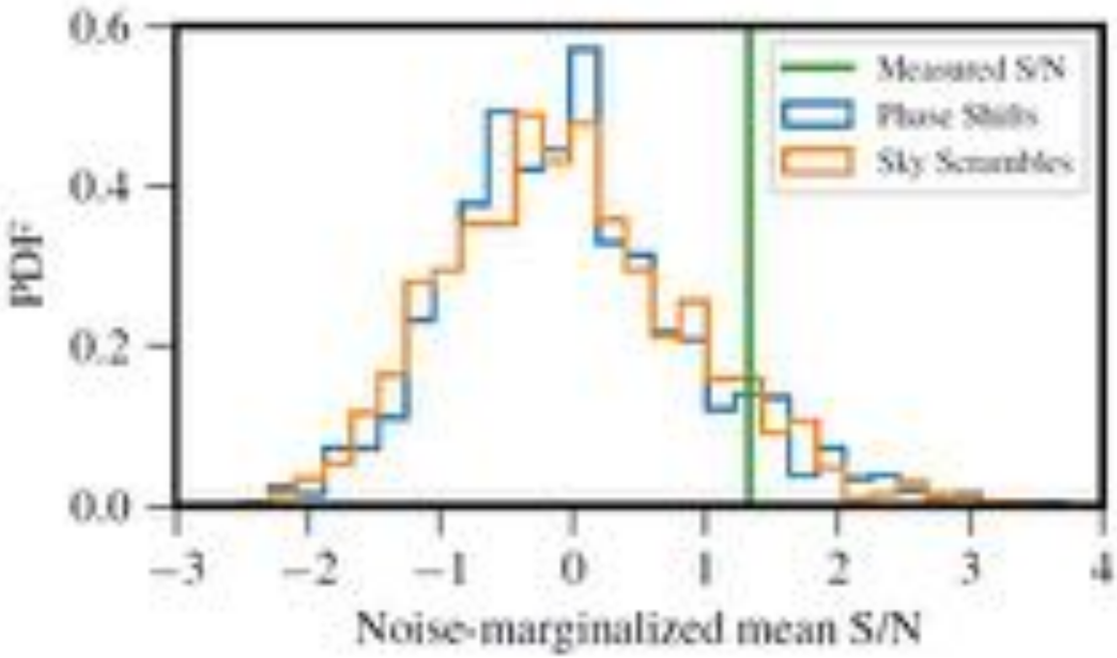


Figure 10. Distribution of the noise-marginalized optimal statistic mean S/N for 1000 phase shifts (blue curve) and 1000 sky scrambles (orange curve). The vertical green line marks the mean S/N measured in the unperturbed model. Higher mean values of the S/N are obtained in 91 phase shifts ($p = 0.091$) and 82 sky scrambles ($p = 0.082$).

Prospects: The way to detect stochastic gravitational wave background

1. Stagnation of improvement in upper limits.
2. Emergence of spatially uncorrelated common-spectrum red process to all the pulsars.
3. Quadrupolar signature is seen in the spatial correlation obtained by Hellings & Downs.

We have reached 2?